A counterexample for the generalized Oort Conjecture

Terezakis Alexios

25 May 2025

National Kapodistrian University of Athens

Aknowledgements The research project is implemented in the framework of H.F.R.I Call "Basic research Financing (Horizontal support of all Sciences)" under the National Recovery and Resilience Plan "Greece 2.0" funded by the European Union Next Generation EU(H.F.R.I. Project Number: 14907. 16618 (MIS: 5163923)).

Introduction

Introduction

We will discuss the (generalized) Oort conjecture, which concerns the problem of lifting algebraic curves with nontrivial automorphisms from characteristic $p > 0$ to characteristic zero.

Setup

- $k =$ algebraically closed field of characteristic $p > 0$,
- $R =$ local ring of characteristic 0, such that $R/\mathfrak{m} \simeq k$,
- $K =$ the quotient field of R.
- $X =$ a curve over k

Witt Vectors

Consider a field k of characteristic $p > 0$. The ring of Witt vectors $W(k)$ is a local ring of characteristic 0 with residue field k, i.e.

$$R/\mathfrak{m} \simeq k.$$

Example
$W(\mathbb{F}_p) = \mathbb{Z}_p$, i.e. the Witt ring of the finite field with p elements is the ring of p-adic integers.

More precisely we will use an extension of the Witt ring, in which we have added the p^h-roots of unity.

Lifting problem

It is well known that any smooth projective algebraic curve X defined over a field of positive characteristic can be lifted to characteristic zero. Now, suppose that the curve is equipped with a subgroup of its automorphism group.

Lifting problem

A question arises: can this group action also be lifted along with the curve to characteristic zero?

This is not always possible since to the Hurwitz bound:

$$|\mathrm{Aut}(X)| \leq 84(g_x - 1),$$

fails in positive characteristic.

When the order of the group is prime to the characteristic this is always possible.

So the interesting case is when the order of G is divisible by p.

Local-global principle

J. Bertin and A. Mezard proved a local global theorem:
(Inv. math. 2000)

Theorem

$$0 \to H^1(X/G, \pi_*^G(\mathcal{T}_X)) \to H^1(G, X, \mathcal{T}_X) \to \bigoplus_{i=1}^{r} H^1\left(G_{x_i}, \widehat{\mathcal{T}}_{X,x_i}\right) \to 0$$

Where $x_1, \ldots, x_r \in X$ are the ramification points, G_{x_i} are the groups that stabilize those points and the completion of the tangent space is $\widehat{\mathcal{T}}_{X,x_i} = k[[t_i]]\frac{d}{dt_i}$.

Local-global principle

According to this theorem, in order to lift a curve with its automorphisms, it suffices to deform a representation over formal power series,

$$\rho : G \hookrightarrow \mathrm{Aut}(k[[t]]).$$

We will call such an action a local action.

Definition
A group G for which every local action can be lifted to characteristic zero is called a **local Oort group**.

Harbater-Katz-Gabber

First, we saw how a local action arises from a curve. Using an HKG-cover, we achieve the reverse: starting from a local action, we construct a curve with a single ramified point, from which the original local action can be recovered.

Definition
Consider a group G with an action on the complete local ring $k[[t]]$. The Harbater–Katz–Gabber compactification theorem asserts that there is a Galois cover $X_{HKG} \to \mathbb{P}^1$ ramified at one point P of X with Galois group $G = \mathrm{Gal}(X_{HKG}/\mathbb{P}^1) = G_0$ and the action of G_0 on the complete local ring $\hat{\mathcal{O}}_{X_{HKG},P}$ coincides with the original action of G_0 on $k[[t]]$.

Oort conjecture

The **Oort conjecture** states that every smooth projective curve over the field k, together with an action of a finite cyclic p-group C_{p^h}, of order p^h, can be lifted to characteristic zero, that is, there exist a lift of both the curve and the action.

Equivalent, every finite cyclic p-group C_{p^h} is a local Oort group.

This conjecture was proven by F. Pop (Ann. of Math. 2014) based on a paper of Obus and Wewers (Ann. of Math. 2014).

Known Results

There are many obstructions for this problem (Bertin-obstruction, **KGB**-obstruction, Hurwitz tree obstruction etc.) and as result we have that:

- The alternating group A_4 is a local Oort group (proven by Obus, (Algebra Number Theory 2016)).
- Dihedral groups D_p of order $2p$. (proven by Bouw, Wewers Duke Math J.))
- Several other cases of D_{p^h} are known: H. Dang, S. Das, K. Karagiannis, A. Obus, V. Thatte, D_4 (B. Weaver)

Generalized Oort conjecture

Another important result is that the KGB obstruction vanishes for all dihedral groups D_{p^h}, where p is an odd prime.

This led to the formulation of the **generalized Oort conjecture**, which asserts that the dihedral groups D_{p^h} are indeed local Oort groups.

Aim of the talk

- To present a new obstruction for groups $G = C_{p^h} \rtimes C_m$.
- To construct a counterexample to the generalized Oort conjecture.

Petri's Theorem

Petri's theorem

Theorem
Let X be a curve over k. There is a small exact sequence,

$$0 \to I_X \to \mathrm{Sym} H^0(X, \Omega_X) \to \bigoplus_{n=0}^{\infty} H^0(X, \Omega_X^{\otimes n}) \to 0,$$

where the ideal I_X generated by elements of degree 2 and 3. Furthermore if X is not non-singular quintic of genus 6 or X is not trigonal curve, then I_X generated by elements of degree 2.

Petri's theorem gives us a description of the canonical embedding of a curve in the projective space by quadratic polynomials.

Relative canonical embedding

In our work, we prove Petri's theorem over Artin local rings. Our proof is based on the following relative version of Petri's theorem (proven by H. Charalambous, K. Karagiannis and A. Kontogeorgis, Annales inst. Fourier 2023).

Theorem
For rings with special and generic fibre, in the next diagram every row is exact and each square is commutative. Moreover, the ideal $I_{\mathcal{X}}$ can be generated by elements of degree 2 as an ideal of S_R.

Relative canonical embedding

$$\begin{tikzcd}
0 \arrow[r] & I_{\mathcal{X}_\eta} \arrow[r, hook] & S_L := L[\omega_1, \ldots, \omega_g] \arrow[r, two heads, "\phi_\eta"] & \bigoplus_{n=0}^{\infty} H^0(\mathcal{X}_\eta, \Omega^{\otimes n}_{\mathcal{X}_\eta/L}) \arrow[r] & 0 \\
0 \arrow[r] & I_{\mathcal{X}} \arrow[r, hook] \arrow[u, hook, "\otimes_R L"'] \arrow[d, two heads, "\otimes_R R/\mathfrak{m}"'] & S_R := R[W_1, \ldots, W_g] \arrow[r, two heads, "\phi"] \arrow[u, hook, "\otimes_R L"'] \arrow[d, two heads, "\otimes_R R/\mathfrak{m}"'] & \bigoplus_{n=0}^{\infty} H^0(\mathcal{X}, \Omega^{\otimes n}_{\mathcal{X}/R}) \arrow[r] \arrow[u, hook, "\otimes_R L"'] \arrow[d, two heads, "\otimes_R R/\mathfrak{m}"'] & 0 \\
0 \arrow[r] & I_{\mathcal{X}_0} \arrow[r, hook] & S_k := k[w_1, \ldots, w_g] \arrow[r, two heads, "\phi_0"] & \bigoplus_{n=0}^{\infty} H^0(\mathcal{X}_0, \Omega^{\otimes n}_{\mathcal{X}_0/k}) \arrow[r] & 0
\end{tikzcd}$$

Where $I_{\mathcal{X}_\eta} = \ker \phi_\eta$, $I_{\mathcal{X}} = \ker \phi$ and $I_{\mathcal{X}_0} = \ker \phi_0$.

Embedded Deformations

Embedded Deformations

Definition
Let Z be a scheme over k and let X be a closed subscheme of Z. An embedded deformation $X' \to \mathrm{Spec} k[\epsilon]$ of X over $\mathrm{Spec} k[\epsilon]$ is a closed subscheme $X' \subset Z' = Z \times \mathrm{Spec} k[\epsilon]$, fitting in the diagram:

$$
\begin{tikzcd}
 & Z \arrow[rr] \arrow[dd] & & Z \times \mathrm{Spec} k[\epsilon] \arrow[dd] \\
X \arrow[ur, hook] \arrow[rr] \arrow[dr] & & X' \arrow[ur, hook] \arrow[dr] & \\
 & \mathrm{Spec} k \arrow[rr] & & \mathrm{Spec} k[\epsilon]
\end{tikzcd}
$$

Embedded Deformations

We fix $Z = \mathbb{P}^{g-1}$ and consider the canonical embedding $X \subset \mathbb{P}^{g-1}$. Assume we have a deformation $X_\Gamma \to \mathrm{Spec}\ \Gamma$ of X, over a local Artin k-algebra Γ, $X_\Gamma \subset \mathbb{P}^{g-1}_\Gamma = \mathbb{P}^{g-1} \times \mathrm{Spec}\ \Gamma$.

By applying the relative version of Petri's theorem, we find that in this case, deformations and embedded deformations are essential the same thing. We have a cohomological proof for this result.

Embedded Deformations

Our initial curve X is described by the homogeneous canonical ideal I_X, which is generated by quadratic polynomials.

The deformations X_Γ of X can also be described by the ideal I_{X_Γ}, which by the relative Petri theorem, is generated by quadratic polynomials with coefficients in Γ.

Every quadratic polynomial in n variables correspond to a $n \times n$ symmetric matrix.

Example

$$a_1x^2 + a_2xy + a_3y^2 = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a_1 & a_2/2 \\ a_2/2 & a_3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Hence, the information of the curve has been fully translated into terms of linear algebra. In particular, the canonical ideal I_X of the initial curve X, is generated by the elements $\{wA_1w^t, \dots, wA_rw^t\}$, where A_i are $n \times n$ symmetric matrices and $w = \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix}$. Similarly the ideal I_{X_Γ} of the relative curve can be described by symmetric matrices with entries in the local Artin ring Γ.

Automorphisms

Curves with Automorphisms

There is a natural action of $\mathrm{Aut}X$ on $H^0(X, \Omega_X)$, which gives rise to a representation,

$$\rho : G \to \mathrm{GL}(H^0(X, \Omega_X)).$$

An element of $\mathrm{GL}(H^0(X, \Omega_X))$ correspond to an automorphism of the curve if and only if respect the canonical ideal I_X.

Criterion

Theorem (Kontogeorgis, T.) - J. Pure Appl. Algebra (accepted)
If there exists a lift of the representation ρ

$$\begin{tikzcd} & \mathrm{GL}_g(R) \arrow[d] \\ G \arrow[ur, "\rho'"] \arrow[r, "\rho"'] & \mathrm{GL}_g(k) \end{tikzcd}$$

which respects the canonical ideal

$$\rho'(\sigma) I_{X_R} = I_{X_R} \quad \text{for all } \sigma \in G,$$

then there is a lift of the curve together with the automorphism group.

Representations

Representations

Definition
A group which is the semidirect product of two cyclic groups $G = C_{p^h} \rtimes C_m$, is called metacyclic, we will treat the case where $(p, m) = 1$.

The dihedral group D_q of order $2q$ is a metacyclic group with $m = 2$.
We begin by describing the indecomposable kG-modules, in order to approach the lifting problem for representations.

Representations

It is known that the indecomposable modules of the cyclic group C_q are determined by the canonical Jordan form of its generator τ. Namely for each $1 \leq k \leq p^h$ there is exactly one indecomposable C_q-module.

$$\begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ 1 & 1 & \dots & & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & 1 & 0 \\ 0 & \dots & \dots & 1 & 1 \end{pmatrix}$$

We can prove there exists an element e such that the following set forms a basis for this module:

$$\{e, (\tau - 1)e, \dots, (\tau - 1)^{k-1}e\}.$$

Representations

In order to extend the action in the whole group $C_q \rtimes C_m$ it suffices to determine how the generator σ of C_m acts on the element e.

$$\sigma e_i = \alpha^{i-1} \zeta_m^{\lambda} e_i + \sum_{\nu=i+1}^{\kappa} a_\nu e_\nu,$$

Additionally, it follows that the eigenvalues of σ are:

$$\{\zeta_m^{\lambda}, \alpha\zeta_m^{\lambda}, \alpha^2\zeta_m^{\lambda}, \dots\}$$

Definition

Let $V_\alpha(\lambda, k)$ be the indecomposable G-module of dimension k given by the base

$$\{(\tau - 1)^{\nu} e, \nu = 0, \dots k-1\}$$

where $\sigma e = \zeta_m^{\lambda} e$.

On the other hand over a local PID ring with residue field k, the form of an indecomposable module is

$$\begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ \alpha_1 & \lambda_2 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \alpha_{d-1} & \lambda_d \end{pmatrix},$$

where λ_i are p^h-roots of unity.

Remark

An indecomposable module over R may decompose into several indecomposable modules after reduction to the residue field.

Liftings of Representations

The theorem bellow is a criterion that allows us to decide whether a representation of $G = C_q \rtimes C_m$ can be lifted or not.

Theorem (Kontogeorgis, T.) - J. Algebra
Consider a $k[G]$-module M which is decomposed as a direct sum

$$M = V_\alpha(\epsilon_1, \kappa_1) \oplus \cdots \oplus V_\alpha(\epsilon_s, \kappa_s).$$

The module lifts to an $R[G]$-module if and only if the set $\{1, \ldots, s\}$ can be written as a disjoint union of sets I_ν, $1 \leq \nu \leq t$ so that

1. $\sum_{\mu \in I_\nu} \kappa_\mu \leq q$, for all $1 \leq \nu \leq t$.
2. $\sum_{\mu \in I_\nu} \kappa_\mu \equiv 0 \text{ or } 1 \mod m$ for all $1 \leq \nu \leq t$.
3. For each ν, $1 \leq \nu \leq t$ there is an enumeration $\sigma : \{1, \ldots, \#I_\nu\} \to I_\nu \subset \{1, .., s\}$, such that

$$\epsilon_{\sigma(2)} = \epsilon_{\sigma(1)} \alpha^{\kappa_{\sigma(1)}}, \epsilon_{\sigma(3)} = \epsilon_{\sigma(3)} \alpha^{\kappa_{\sigma(3)}}, \ldots, \epsilon_{\sigma(s)} = \epsilon_{\sigma(s-1)} \alpha^{\kappa_{\sigma(s-1)}}$$

Remark

1. The first condition follows from the fact that modular representations of the cyclic group C_q have dimension $= 1, \ldots, q$ (where $q =$ is a power of $\mathrm{char}(R)$).
2. For the second condition just note that the eigenvectors of τ split into orbits of σ.
3. We will explain the third condition with an example.

Example

Consider the group $C_{25} \rtimes C_4$ ans $\alpha = 7$. The module $V(1,2) \oplus V(3,2)$ lift to characteristic zero, where the matrix T with respect to the base e_1, e_2, e_3, e_4 is given by

$$T = \begin{pmatrix} \zeta_q & 0 & 0 & 0 \\ 1 & \zeta_q^2 & 0 & 0 \\ 0 & \pi & \zeta_q^3 & 0 \\ 0 & 0 & 1 & \zeta_q^4 \end{pmatrix}$$

and the eigenvalues of σ are

$$\{\zeta_4, \alpha\zeta_4, \alpha^2\zeta_4, \alpha^3\zeta_4\} = \{\zeta_4, \zeta_4^2, \zeta_4^3, 1\}$$

since $\alpha = 7 = \zeta_4$. Observe that the matrix T is indecomposable in R but not modulo $\mathfrak{m}$.

Decomposition of $H^0(X, \Omega_X)$

In order to construct the counterexample, we will need to count how many times each indecomposable summand appears in the G-module

$$H^0(X, \Omega_X).$$

We will rely on the work of F. Bleher, T. Chinburg, and A. Kontogeorgis (J. Number Theory, 2020), in which they provided a formula for the multiplicity of each indecomposable summand.

Let us also mention that this formula depends on lower jumps of the action.

Lower jumps

In order to study the local actions in characteristic $p > 0$, we introduce the notion of ramification filtration of a group G on each point $x \in X$. Let G be a subgroup of $\mathrm{Aut}(X)$ and let $\mathfrak{m}_{X,x}$ be the maximal ideal of the local ring $\mathcal{O}_{X,x}$. For $i \geq 0$, the i-th ramification group Gx, i of G in x is the subgroup of the elements $\sigma \in G$ that stabilize x and act trivially on $\mathcal{O}_{X,x}/\mathfrak{m}_{X,x}^{i+1}$. Those groups form a descending finite filtration

$$G_{x,0} \rhd G_{1,x} \rhd \cdots G_{n,x} \rhd G_{n+1,x} = 1.$$

We call i a lower jump when in the filtration

$$G_{i,x} \supsetneq G_{i+1,x}.$$

The Counterexample

(Kontogeorgis, T.) - Ann. Sci. École Norm. Sup. (accepted)

Let $G = D_{125}$ with lower jumps $9, 189, 4689$. Below we see the decomposition in indecomposable submodules of $H^0(X, \Omega_X)$:

$\boldsymbol{U_{0,1}}, \boldsymbol{U_{1,1}}, U_{0,2}, U_{1,2}, \boldsymbol{U_{1,3}}, U_{0,4}, U_{1,4}, U_{0,5}, U_{1,6}, U_{0,7}, U_{1,7}, U_{0,8}, U_{1,8},$
$U_{0,9}, U_{1,9}, U_{0,11}, U_{1,11}, U_{0,12}, U_{1,12}, U_{0,13}, U_{1,13}, U_{0,14}, U_{1,15}, U_{0,16},$
$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
$U_{0,112}, U_{1,112}, U_{0,113}, U_{1,113}, U_{0,114}, U_{1,115}, U_{0,116}, U_{1,116}, U_{0,117}, U_{0,118},$
$U_{1,118}, U_{0,119}, U_{1,119}, \boldsymbol{U_{0,121}}, \boldsymbol{U_{1,121}}, U_{0,122}, U_{1,122}, \boldsymbol{U_{0,123}}, \boldsymbol{U_{1,123}}, U_{1,124}.$

The sum of the dimension cannot by greater than 125, hence the indecomposable summands $U_{0,123}, U_{1,123}$ must glue with $U_{0,1}, U_{1,1}$. The modules $U_{0,121}, U_{1,121}$ cannot be combined both, since there is only one module with dimension less than 3, the $U_{1,3}$.

Thank you!