Ihara Theory on the Heisenberg Covering Tower of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$

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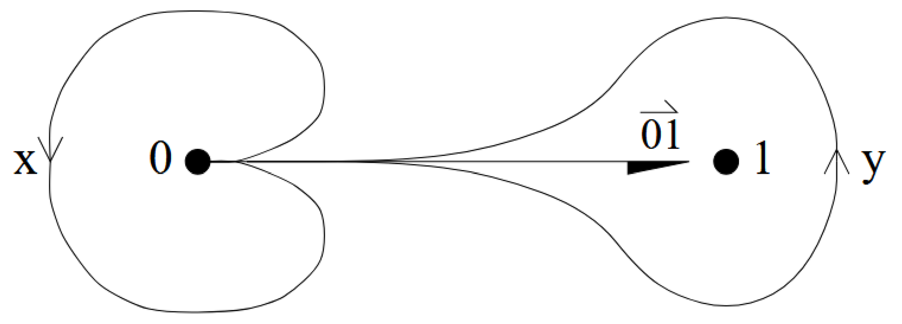
Introduction

Let $X = \mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$ be the projective t-line over $\mathbb{Q}$ minus the three points and pick the tangential basepoint $\vec{01} : \mathrm{Spec}\mathbb{Q}((\mathrm{t})) \to X$, associated to the embedding $\mathbb{Q}(t) \hookrightarrow \mathbb{Q}((t))$. After fixing $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and an odd prime number ℓ, the ℓ-component of the geometric fundamental group $\pi_1(X_{\overline{\mathbb{Q}}}, \vec{01})$ is identified with the pro-ℓ completion of the topological fundamental group $\pi_1^{\mathrm{top}}(X_{\mathbb{C}}, \vec{01})$.

Denote the ℓ-component of $\pi_1(X_{\overline{\mathbb{Q}}}, \vec{01})$ by $\mathcal{F}$. It is free profinite on two generators x, y representing the classes of loops around $0, 1$ and admits a presentation $\mathcal{F} = \langle x, y, z \mid xyz = 1 \rangle$.



The exact sequence of profinite groups

$$1 \longrightarrow \mathcal{F} \longrightarrow \pi_1(\mathbb{P}^1_{\overline{\mathbb{Q}}} - \{0, 1, \infty\}, \vec{01}) \longrightarrow \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow 1$$

gives rise to the Galois representation $\phi_{\vec{01}}$:

$$\phi_{\vec{01}} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{Aut}(\mathcal{F})$$

$$\sigma \longmapsto \left(x \mapsto x^{\chi_\ell(\sigma)}, \quad y \mapsto f_\sigma^{-1}\, y^{\chi_\ell(\sigma)}\, f_\sigma \right)$$

with $\chi_\ell : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}_\ell$ being the cyclotomic character and $f_\sigma \in \mathcal{F}' = [\mathcal{F}, \mathcal{F}]$ representing $[\mathrm{path} : 0 \to 1]^{-1} \cdot \sigma([\mathrm{path} : 0 \to 1])$.

Towers of Curves

Assume a pro-ℓ exact sequence where $\mathcal{N}$ is a closed subgroup of $\mathcal{F}$

$$1 \longrightarrow \mathcal{N} \longrightarrow \mathcal{F} \xrightarrow{\pi} \mathcal{G} \longrightarrow 1$$

If $\{Y_n/\mathbb{P}^1\}_{n=1}^{\infty}$ are finite Galois covers of $\mathbb{P}^1$ unramified outside $0, 1, \infty$ such that $\mathcal{G} = \varprojlim \mathrm{Gal}(Y_n/\mathbb{P}^1)$ then there is the natural identification

$$\mathcal{N}/\mathcal{N}' \simeq \varprojlim T_\ell(\mathrm{Jac}(Y_n)) \simeq \varprojlim H_1(Y_n, \mathbb{Z}_\ell)$$

Blanchfield-Lyndon Theorem

The profinite version of Fox Free Differential calculus developed by Ralph Fox utilizes continuous, $\mathbb{Z}_\ell$-linear mappings

$$\frac{\partial}{\partial x_i} : \mathbb{Z}_\ell[[\mathcal{F}]] \to \mathbb{Z}_\ell[[\mathcal{F}]]$$

$$\frac{\partial x_j}{\partial x_i} = \delta_{ij}, \quad \frac{\partial ab}{\partial x_i} = \frac{\partial a}{\partial x_i} + a\frac{\partial b}{\partial x_i}$$

The Blanchfield-Lyndon theorem asserts that

$$\mathcal{N}/\mathcal{N}' \simeq \{(\xi, \eta) \in \mathbb{Z}_\ell[[\mathcal{G}]]^2 : \ \xi(\pi(x) - 1) + \eta(\pi(y) - 1) = 0\}$$

as $\mathbb{Z}_\ell[[\mathcal{G}]]$-modules! via the mapping $w \in \mathcal{N}/\mathcal{N}' \mapsto \left(\pi\frac{\partial w}{\partial x}, \pi\frac{\partial w}{\partial y}\right)$.

Examples: Fermat and Heisenberg Curves

- If $\mathcal{N} = \mathcal{F}'$, then $\mathcal{G} = \mathbb{Z}_\ell \times \mathbb{Z}_\ell$ and each Y_n curve has group $\mathrm{Gal}(Y_n/\mathbb{P}^1) = \mathbb{Z}/\ell^n\mathbb{Z} \times \mathbb{Z}/\ell^n\mathbb{Z}$. This can be realized by the subgroup $\mathcal{F}' \cdot \langle x^{\ell^n}, y^{\ell^n}, z^{\ell^n} \rangle$, which corresponds to the function field $\overline{\mathbb{Q}}(t, t^{1/\ell^n}, (t-1)^{1/\ell^n})$ with Y_n being its non-singular projective model, the Fermat curve $X^{\ell^n} + Y^{\ell^n} = Z^{\ell^n}$.
- If $\mathcal{N} = [\mathcal{F}, \mathcal{F}'] = \mathcal{F}(3)$ as in the lower central series, then $\mathcal{G}$ is the pro-ℓ Heisenberg group:

$$\mathcal{H}_\ell := \begin{pmatrix} 1 & \mathbb{Z}_\ell & \mathbb{Z}_\ell \\ 0 & 1 & \mathbb{Z}_\ell \\ 0 & 0 & 1 \end{pmatrix}$$

and since $[\mathcal{F}, \mathcal{F}'] \leq \mathcal{F}'$ each Heisenberg curve Y_n is a covering of the ℓ^n-level Fermat curve, unramified outside $0, 1, \infty$ with group $\mathrm{Gal}(Y_n/\mathbb{P}^1) = \mathcal{H}_\ell \bmod \ell^n$.

Theorem 1 [Ihara]

The Galois action on $\mathcal{N}/\mathcal{N}'$ induces a cocycle

$$\psi : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathbb{Z}_\ell[[\mathcal{G}]]^\times, \quad \psi(\sigma) = \pi\left(f_\sigma + \frac{\partial f_\sigma}{\partial y}(y-1)\right)$$

Theorem 2 [Ihara]

The group $\mathcal{F}'/\mathcal{F}''$ is a free $\mathbb{Z}_\ell[[\mathbb{Z}_\ell \times \mathbb{Z}_\ell]]$-module of rank 1, generated by $[x, y]$.

The action via $\phi_{\vec{01}}$ induces the Ihara power series F_σ in $\mathbb{Z}_\ell[[\mathbb{Z}_\ell \times \mathbb{Z}_\ell]]$ by $\sigma \cdot [x, y] = F_\sigma \cdot [x, y]$. The power series F_σ are of high number theoretic interest, its specializations are related to the Vandiver conjecture, and they coincide with $\psi(\sigma)$ for $\mathcal{N} = \mathcal{F}'$ up to some units.

Goal

Understand $\psi(\sigma)$ regarding the Heisenberg tower and introduce a non-commutative Heisenberg analogue of F_σ.

Result/Obstruction

The $\mathbb{Z}_\ell[[\mathcal{H}_\ell]]$-module $\mathcal{F}(3)/\mathcal{F}(3)'$ is neither free nor cyclic.

Related result regarding each Heisenberg curve

Let Y_n be the Heisenberg curve as above, K a field of characteristic 0 containing the ℓ^n-th roots of unity and $G = \mathcal{H}_\ell \bmod \ell^n$. Using a generalized version of the Blanchfield-Lyndon Theorem, the Crowell exact sequence, the first homology group $H_1(Y_n, K)$ has been classified as a $K[G]$-module.

References

[1] Jannis A. Antoniadis and Aristides Kontogeorgis. The group of automorphisms of the Heisenberg curve. In *Abelian Varieties and Number Theory*, volume 767 of *Contemp. Math.*, pages 25–39. Amer. Math. Soc., 2021.

[2] Yasutaka Ihara. Braids, Galois groups, and some arithmetic functions. *in Proc. ICM90(I), pp. 99–120 (Mathematical Society of Japan, Tokyo, 1991)*.

[3] Yasutaka Ihara. On Galois representations arising from towers of coverings of $\mathbf{P}^1\{0, 1, \infty\}$. *Invent. Math.*, 86(3):427–459, 1986.

[4] Yasutaka Ihara. Profinite braid groups, Galois representations and complex multiplications. *Ann. of Math. (2)*, 123(1):43–106, 1986.

[5] Aristides Kontogeorgis and Dimitrios Noulas. Galois action on homology of the Heisenberg curve. *arxiv.org/abs/2411.11140*, 2024.