Descent and Equivariant Categories

Galois Actions on Algebraic Varieties

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Introduction

Mathematicians have been studying descent of algebraic varieties for a long time. This journey began with Weil's work [15], Grothendieck generalized it in [8] using fibered categories and finally Mumford's Geometric Invariant Theory [11] gave it a huge boost which partly led to the development of the theory of stacks. Following the definition of G-linearizations of invertible sheaves in [11] (for this work G is an algebraic group), there is a simplified version for finite groups which yields the definition of equivariant objects of a category with a G-action (Definition 3, see also [4] for a topological group analogue). Equivariant categories have their roots in Grothendieck's seminal Tôhoku paper [7] where he studied the category of G-equivariant sheaves of a topological space. Later Mumford in [12] discovered an equivalence between the equivariant category of coherent sheaves and the quotient variety (see Example 4 below). Since then, equivariant categories of sheaves on a topological space have been studied as a non-commutative categorical quotient for which the exploration of geometric properties and structure is very appealing (see [3] and [6]). The theory of equivariant (derived) categories is still being developed and is very relevant to many mathematical fields such as algebraic geometry, representation theory, descent theory and the Langlands program. [1], [14].

Aim

Our aim is to take advantage of the equivariant category theory to provide a different point of view for the descent of algebraic varieties, especially for the Galois descent in the spirit of Weil's work [15] and to bridge together the two well-known theories which seem to be deeply related. To achieve this we need to understand and define a suitable Galois action on the category of coherent sheaves of an algebraic variety and show that the quotient variety is Weil's descent variety.

Ingredient 1 - Galois Descent for Algebraic Varieties

Basic Idea. Let K be a field of positive characteristic and K_s a fixed separable closure. Then $\mathrm{Gal}(K_s, K)$ acts on polynomials defined over K_s by permuting the coefficients and thus naturally yields an action on algebraic varieties over K_s.

Let $p\colon X \to \mathrm{Spec}(K_s)$ be a K_s-scheme. For every $\sigma \in \mathrm{Gal}(K_s, K)$ the ring homomorphism $\sigma\colon K_s \to K_s$ (contravariantly) corresponds to a morphism of schemes $\tilde{\sigma}\colon \mathrm{Spec}(K_s) \to \mathrm{Spec}(K_s)$. Then we can define the action of σ on X as the fiber product ${}^{\sigma}X := X \times_{\tilde{\sigma}} \mathrm{Spec}(K_s)$ with $\pi_2\colon {}^{\sigma}X \to \mathrm{Spec}(K_s)$.

$$\begin{tikzcd} X \arrow[d, "p"'] & {}^{\sigma}X \arrow[l, "\pi_1"'] \arrow[d, "\pi_2"] \\ \mathrm{Spec}(K_s) & \mathrm{Spec}(K_s) \arrow[l, "\tilde{\sigma}"] \end{tikzcd}$$

Note that every variety is defined over a field L which is a finite extension K, i.e. the ideal $I(X)$ can be generated by a finite collection of L-polynomials.

Definition 1. Let L be a field such that $K \subset L \subset K_s$ and assume that X is defined over L. We say that X is definable over K with respect to the Galois extension L/K if there is an algebraic variety Y_0 defined over K and an isomorphism $R\colon X \to Y_0 \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)$ defined over L. The smallest field L such that X is definable over K is called the *field of definition*.

Assume X defined over L and definable over K with isomorphism $R\colon X \to Y$ over L, where $Y := Y_0 \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)$. Then there are isomorphisms ${}^{\sigma}R\colon {}^{\sigma}Y \to {}^{\sigma}X$ such that the family of isomorphisms

$$\{f_\sigma = ({}^{\sigma}R)^{-1} \circ (\mathrm{Id}_Y \times \sigma) \circ R\colon X \to {}^{\sigma}X\}_{\sigma\in\mathrm{Gal}(L/K)}$$

satisfies the condition

$$f_{\sigma\tau} = {}^{\sigma}f_\tau \circ f_\sigma \quad \forall \sigma, \tau \in \mathrm{Gal}(L/K)$$

Diagrammatically we have the following:

$$\begin{tikzcd} & {}^{\tau}Y \arrow[rr, "({}^{\tau}R)^{-1}"] & & {}^{\tau}X & \\ {}^{\sigma\tau}Y \arrow[rrrr, bend left=40, "({}^{\sigma\tau}R)^{-1}"] & Y \arrow[l, "\mathrm{Id}_Y\times\sigma\tau"'] \arrow[u, "\mathrm{Id}_Y\times\tau"] \arrow[d, "\mathrm{Id}_Y\times\sigma"'] & X \arrow[l, "R"'] \arrow[ur, "f_\tau"] \arrow[dr, "f_\sigma"'] \arrow[rr, "f_{\sigma\tau}"] & & {}^{\sigma\tau}X \\ & {}^{\sigma}Y \arrow[ul, bend left, "{}^{\sigma}(\mathrm{Id}_Y\times\tau)"] \arrow[rr, "({}^{\sigma}R)^{-1}"] & & {}^{\sigma}X \arrow[ur, bend right, "{}^{\sigma}f_\tau"'] & \end{tikzcd}$$

Such a family $\{f_\sigma\}_{\sigma\in\mathrm{Gal}(L/K)}$ is called a *Galois descent datum* of X with respect to L/K.

Definable over $K \Rightarrow$ Admits Galois descent datum

The inverse also holds:

Weil's Descent Theorem. Consider the Galois extension L/K, where $K \subset L \subset K_s$.

1. If X admits a Galois descent datum $\{f_\sigma\}_{\sigma\in\mathrm{Gal}(L/K)}$ with respect to L/K, then there exists an algebraic variety Y defined over K and an isomorphism $R : X \to Y$ defined over L, such that $R = {}^{\sigma}R \circ f_\sigma$ for every $\sigma \in \mathrm{Gal}(L/K)$.
2. If there is another variety $\widehat{Y}$ defined over K and an isomorphism $\widehat{R}\colon X \to \widehat{Y}$, defined over L such that $\widehat{R} = {}^{\sigma}\widehat{R} \circ f_\sigma$ for all $\sigma \in \mathrm{Gal}(L/K)$, then there exist an isomorphism $J : Y \to \widehat{Y}$, defined over K such that $\widehat{R} = J \circ R$.

Ingredient 2 - Categories

Category theory was developed a lot by Grothendieck in order to provide a unified language and a framework for his theory regarding sheaves of schemes, or generally of topological spaces. Since then, mathematicians have found that categories are great invariants of schemes and have even reconstructed schemes using suitable categories. Namely, let X be a smooth projective variety, then the following categories reconstruct X as a ringed space:

- $\mathsf{Coh}(X)$, originally by Gabriel for Noetherian schemes, for quasi-separated by Rosenberg, general case by Calabrese, Pirisi
- $\mathsf{D}^{\mathsf{b}}(\mathsf{Coh}(X))$ when ω_X is (anti-)ample, by Bondal & Orlov [5]
- $\mathsf{D}^{\mathsf{b}}(\mathsf{Coh}(X))$ with its monoidal structure, by Balmer [2]

Equivariant Categories

Definition 2. Let G be a finite group and $\mathcal{A}$ an additive category. A group action of G on $\mathcal{A}$ consists of the following data:

(i) $\forall g \in G$ an auto-equivalence $g\colon \mathcal{A} \to \mathcal{A}$,

(ii) a family of natural isomorphisms $\theta_{g,h}\colon g \circ h \xrightarrow{\simeq} (gh)$

(iii) $\forall g, h, k$ commutativity of:

$$\begin{tikzcd} g\circ h\circ k \arrow[r] \arrow[d] & g\circ(hk) \arrow[d] \\ (gh)\circ k \arrow[r] & (ghk) \end{tikzcd}$$

Definition 3. Given the data of a group action on an additive category $\mathcal{A}$, the *equivariant category* $\mathcal{A}^G$ has:

Objects: (E, ϕ) where $E \in \mathsf{Ob}(\mathcal{A})$ and $\{\phi_g : E \xrightarrow{\simeq} gE\}_{g\in G}$ is a family of isomorphisms called the *linearization* of E, satisfying the following commutative diagram:

$$\begin{tikzcd} E \arrow[r, "\phi_g"] \arrow[rrr, bend right, "\phi_{gh}"'] & gE \arrow[r, "g\phi_h"] & ghE \arrow[r, "\theta_{g,h}"] & (gh)E \end{tikzcd}$$

&

Morphisms: $f\colon (E, \phi) \to (E', \phi')$ are moprhisms of $\mathcal{A}$ making the following diagram commute for all $g \in G$:

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\phi_g"'] & E' \arrow[d, "\phi'_g"] \\ gE \arrow[r, "gf"] & gE' \end{tikzcd}$$

Note that a G-action on an abelian category $\mathcal{A}$ naturally induces a group action on its derived category $\mathsf{D}^{\mathsf{b}}(\mathcal{A})$ and the equivariant category $\mathsf{D}^{\mathsf{b}}(\mathcal{A})^G$ is canonically triangulated as long as $|G|$ is invertible in $\mathcal{A}$ (each $f \in \mathrm{Hom}_{\mathcal{A}}$ is uniquely divisible by $|G|$). Moreover, $\mathcal{A}^G$ is also abelian and it turns out that we have an equivalence of triangulated categories:

$$\mathsf{D}^{\mathsf{b}}(\mathcal{A}^G) \simeq \mathsf{D}^{\mathsf{b}}(\mathcal{A})^G$$

Example 4. Let $G \subset \mathsf{Aut}(X)$ be a finite subgroup. Then every G acts on $\mathsf{Coh}(X)$ by pushforwards g_* along each automorphism $g \in G$. The quotient variety X/G exists if each G-orbit is contained in some affine open [9][Exposé V, Proposition 1.8] and if the action is free then $\mathsf{Coh}^G(X) \simeq \mathsf{Coh}(X/G)$ by Mumford [12, Chapter II, Paragraph 7].

In the level of derived categories, we have triangle equivalences $\mathsf{D}^{\mathsf{b}}(\mathsf{Coh}(X))^G \simeq \mathsf{D}^{\mathsf{b}}(\mathsf{Coh}^G(X)) \simeq \mathsf{D}^{\mathsf{b}}(\mathsf{Coh}(X/G))$.

Combining the Ingredients - Equivalence of Categories with Descent Flavors

- K-Var := category of K-varieties
- L-Var_K := category of L-varieties definable over K

$(\star)$ L-$\mathsf{Var}^{\mathrm{Gal}(L/K)} \simeq L$-$\mathsf{Var}_K \simeq K$-$\mathsf{Var}$

Explanation of the equivalences of $(\star)$

We have that $\mathrm{Gal}(L/K)$ acts on the category L-Var and its equivariant category L-$\mathsf{Var}^{\mathrm{Gal}(L/K)}$ consists of pairs (X, f) where the linearization $\{f_\sigma\}_{\sigma\in\mathrm{Gal}(L/K)}$ is in fact a Galois descent datum.

The equivalence of categories L-$\mathsf{Var}^{\mathrm{Gal}(L/K)} \simeq L$-$\mathsf{Var}_K$ is obtained by the observation that every L-variety is definable over K if and only if admits Galois descent datum.

The functor K-$\mathsf{Var} \to L$-Var_K is given by $Y_0 \mapsto (Y_0 \times \mathrm{Spec}(L), \{\mathrm{Id} \times \sigma\}_{\sigma\in\mathrm{Gal}(L/K)})$ and its quasi-inverse is given by Weil's Descent Theorem.

Galois Actions on Categories of Sheaves

Let L/K be a finite Galois field extension and consider a L-variety X. We would like to define a $\mathrm{Gal}(L/K)$-action on X such that we obtain the quotient $X/\mathrm{Gal}(L/K) = Y$ where Y is defined over K. One way to achieve this is by showing that there is an equivalence of categories $\mathsf{Coh}(X)^{\mathrm{Gal}(L/K)} \simeq \mathsf{Coh}(Y)$ or even $\mathrm{D}^{\mathrm{b}}(X)^{\mathrm{Gal}(L/K)} \simeq \mathrm{D}^{\mathrm{b}}(Y)$ and then using some reconstruction theorem.

Our Attempt - Main Idea

Each $\sigma \in \mathrm{Gal}(L/K)$ induces a homeomorphism $\hat{\sigma}\colon {}^{\sigma}X \to X$ (it is the projection π_1 as described above). Indeed if U is a complement of a closed subscheme V then $\hat{\sigma}(U) = {}^{\sigma}X \setminus {}^{\sigma}V$ is open. Assume that there exists a descent datum $\{f_\sigma\}_{\sigma\in\mathrm{Gal}(L/K)}$.

Proposition 5. *There exists a natural* $\mathrm{Gal}(L/K)$*-action on* X *by homeomorphisms:*

$$\begin{aligned} X &\longrightarrow X \\ P &\longmapsto \hat{\sigma} f_\sigma(P) \end{aligned}$$

Given an open morphism of schemes $\mu\colon X \to Y$ and a sheaf $\mathcal{F}$ of Y we can easily define the inverse image sheaf $\mu^{-1}\mathcal{F}$ of X and the $\mathcal{O}_X$-module $\mu^*\mathcal{F}$ as follows. Given an open $U \subset X$ we define:

$$\mu^{-1}\mathcal{F}(U) = \mathcal{F}(\mu(U)) \qquad \& \qquad \mu^*\mathcal{F}(U) = \mu^{-1}\mathcal{F}(U) \otimes_{\mu^{-1}\mathcal{O}_Y(U)} \mathcal{O}_X(U)$$

Key Point: In this construction the definition of a morphism of schemes (being ringed spaces) provides us with a ring homomorphism $\mu^{-1}\mathcal{O}_Y(U) \to \mathcal{O}_X(U)$ which allows us the computation of the tensor product, see [10, II.5]. Indeed, for any continuous function $\mu\colon X \to Y$ we know that μ^{-1} is a left adjoint of μ_*, that is there is a natural map $\mu^{-1}\mu_*\mathcal{O}_X \to \mathcal{O}_X$, and moreover since the map μ is a morphism of schemes we have $\mu_*\mathcal{O}_X = \mathcal{O}_Y$, [10, Exer. II.1.18].

We will now consider the homeomorphism $\hat{\sigma}\colon {}^{\sigma}X \to X$ and for a sheaf $\mathcal{F}$ of ${}^{\sigma}X$ and an open $U \subset X$ we define

$$\hat{\sigma}^*(\mathcal{F})(U) = \hat{\sigma}^{-1}(\mathcal{F})(U) \otimes_{\hat{\sigma}^{-1}\mathcal{O}_X(U)} \mathcal{O}_{{}^{\sigma}X}(U)$$

where the map $\tilde{\sigma}^{-1}\mathcal{O}_X(U) = \mathcal{O}_X(\sigma U) \to \mathcal{O}_{{}^{\sigma}X}(U)$ is given by $f \in \tilde{\sigma}^{-1}\mathcal{O}_X(U) = \mathcal{O}_X(\sigma(U)) \mapsto {}^{\sigma^{-1}}f \in \mathcal{F}(U)$.

Proposition 6. *Let* $\mathcal{F}$ *be a sheaf of* $\mathcal{O}_X$*-modules on* X*, the map* $\mathcal{F} \mapsto (\tilde{\sigma} f_\sigma)^*\mathcal{F} \simeq f_\sigma^*\tilde{\sigma}^*\mathcal{F}$ *is a* $\mathrm{Gal}(L/K)$*-action on the category of* $\mathcal{O}_X$*-modules of* X*.*

To Do List:

- Construct an equivalence of categories $\mathsf{Coh}(X)^{\mathrm{Gal}(L/K)} \simeq \mathsf{Coh}(Y)$ or $\mathsf{D}^{\mathsf{b}}(X)^{\mathrm{Gal}(L/K)} \simeq \mathsf{D}^{\mathsf{b}}(Y)$.
- Figure out which reconstruction theorem suits our setup.
- If needed, modify the definition of the group action to resolve any issues.

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New Research!

A lot of the theory on equivariant categories is also showcased in our recent article (along with new results) in a joint work with A. Kontogeorgis and C. Psaroudakis: " *Equivariant Recollements and Singular Equivalences* " - arXiv:2504.07620

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