The Heisenberg curve in Arithmetic Topology

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Με τη χρηματοδότηση της Ευρωπαϊκής Ένωσης
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What is Arithmetic Topology?

Mazur, B.: Remarks on the Alexander polynomial. Unpublished Note (1963 or 1964)
Mazur, B.: Notes on étale cohomology of number fields,. Ann. Sci. Éc. Norm. Super. (4) 6, 521–552 (1973)

Mazur-Morishita-Kapranov-Reznikov dictionary

$S^1 \hookrightarrow M$	$\mathrm{Spec}(\mathbb{F}_p) \hookrightarrow \mathrm{Spec}(\mathcal{O}_k)$
Knot	Prime Ideal
Link $S^1 \coprod S^1 \coprod \cdots \coprod S^1$	Ideal $I = \mathfrak{p}_1^{e_1}\mathfrak{p}_2^{e_2}\cdots\mathfrak{p}_r^{e_r} \subset \mathcal{O}_k$
conjugacy class of braids	conj. class of Frobenius elements $x \mapsto x^p$

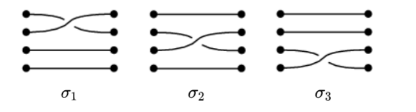
Artin meets Ihara!

Ihara 1986

What is the knot equivalent of Gal($\overline{\mathbb{Q}}/\mathbb{Q}$)?
Profinite braids! *Profinite braid groups, Galois representations and complex multiplications, Ann. of Math. (2) 123 (1986), 43–106.*

Artin representation of braids

$F_{s-1} = \langle x_1, x_2, \ldots, x_{s-1}, x_s \mid x_1 x_2 \cdots x_s = 1 \rangle$ is the free group on $s-1$ generators. The braid group B_{s-1} as a subgroup of $Aut(F_{s-1})$ is generated by the elements σ_i:



$$\sigma_i(x_{i+1}) = x_i,\ \sigma_i(x_i) = x_i x_{i+1} x_i^{-1},\ \sigma_i(x_k) = x_k$$

Artin meets Ihara!

Ihara representation

$$\mathrm{Ih}_s : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{Aut}(\mathfrak{F}_{s-1})$$

where $\mathfrak{F}_{s-1}$ is the pro-ℓ completion of F_{s-1} with image inside the subgroup

$$\{\sigma \in \mathrm{Aut}(\mathfrak{F}_{s-1}) \mid \ \sigma(x_i) \sim x_i^{N(\sigma)},\ N(\sigma) \in \mathbb{Z}_\ell^\times\}$$

where

$$N \circ \mathrm{Ih}_s : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathbb{Z}_\ell^\times$$

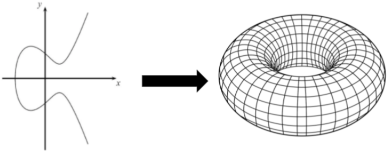
is the cyclotomic character χ_ℓ such that, if $\sigma(\zeta_{\ell^k}) = \zeta_{\ell^k}^{t_k}$ then $\chi_\ell(\sigma) = \varprojlim t_k$.

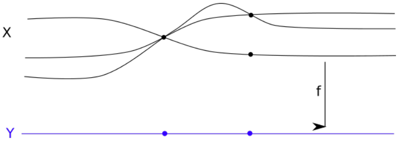
What is an Algebraic curve?

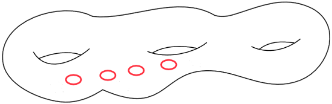
-Everyone knows what a curve is, until he has studied enough mathematics to become confused... - F. Klein

Example

1. $y^2 = x^3 + ax + b$
2. Function field $k(x)(\sqrt{x^3 + ax + b})$
3. Riemann Surface







The Fermat curve

$$Fer_n : \quad x^n + y^n = 1$$

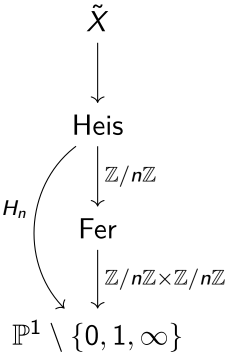
$$\text{Galois group: } \sigma_{ij}(x,y) = (\zeta^i x, \zeta^j y)$$

Heisenberg group modulo n

$$H_n := \left\{ \begin{pmatrix} 1 & i & k \\ 0 & 1 & j \\ 0 & 0 & 1 \end{pmatrix}, \quad i,j,k \in \mathbb{Z}/n\mathbb{Z} \right\}$$

$$\begin{pmatrix} 1 & i & k \\ 0 & 1 & j \\ 0 & 0 & 1 \end{pmatrix} \mapsto \sigma_{ij}$$

The Heisenberg curve



$$F_2 = \langle a, b \rangle = \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\})$$

$$R_{\mathrm{Fer}_n} = \langle a^n, b^n, [a, b] \rangle$$

$$R_{\mathrm{Heis}_n} = \langle a^n, b^n, [a, [a, b]], [b, [a, b]] \rangle$$

$$1$$

$$1 \to R_{\mathrm{Heis}} \to F_2 \to H_n \to 1$$

Well-defined Galois action by conjugation of H_n on R_{Heis}^{ab}

A basis and some Representation theory

Basis of the Homology of the closed curve

for odd n:

$$R_{\mathrm{Heis}_n} = \langle a_1, b_1, \dots, a_g, b_g, \gamma_1, \dots, \gamma_{3n^2} \mid [a_1, b_1] \cdots [a_g, b_g]\gamma_1 \cdots \gamma_{3n^2} = 1 \rangle$$

Let $\Gamma = \langle a^n, b^n, (ab)^n \rangle$, then $H_1(X_H, \mathbb{Z})$ is generated by

$$[a, [a, b]]^{\alpha^i \tau^k} \mod \Gamma, \ [b, [a, b]]^{\alpha^i \tau^j \beta^j} \mod \Gamma$$

Irreducible characters of H_n

$$\chi_{ijs}(g) = \sum_{h \in cl(g)} (\chi_{ij} \otimes \chi_s)(h)$$

$$\alpha \mapsto \zeta^i, \ \tau \mapsto \zeta^j, \ \beta \mapsto \zeta^s$$

$$0 \le i, j, \le n-1, 0 \le s \le \gcd(n, j) - 1$$

More Arithmetic Topology

Alexander Modules

$$\psi : G \to H,\ \ N := \ker\psi,\ \ G = \langle x_1, \dots, x_m \mid R_1, \dots, R_s = 1\rangle$$

$$\text{Alexander module } \mathcal{A}_\psi,\ \ \frac{\partial R_i}{\partial x_j}$$

$\pi_1(M \setminus K_1 \cup \cdots \cup K_r)$	$\pi_1(\mathrm{Spec}(\mathcal{O}_k) \setminus \{p_1, \dots p_r\})$
Link (or Knot) module $\mathbb{Z}[H]$	Iwasawa module $\mathbb{Z}_\ell[[H]]$

Crowell exact sequence

$$1 \to N^{ab} \to \mathcal{A}_\psi \to \mathbb{Z}[H] \to \mathbb{Z} \to 1$$

$$1 \to N^{ab} \to \mathcal{A}_\psi \to \mathbb{Z}_\ell[[H]] \to \mathbb{Z}_\ell \to 1$$

exact sequence of $\mathbb{Z}[H]$-(resp. $\mathbb{Z}_\ell[[H]]$-) modules

Result

Homology of the closed Heisenberg curve as an $\mathbb{F}[H_n]$-module

$$H_1(X_H, \mathbb{F}) = \bigoplus_{i,j=0}^{n-1} \bigoplus_{s=0}^{\gcd(n,j)-1} \mathbb{F} h_{ijs} \chi_{ijs},$$

where

$$h_{ijs} = \begin{cases} 1 - z(i,s) & \text{if } (i,s) \neq (0,0),\ j = 0 \\ \frac{n}{\gcd(n,j)}, & \text{if } (i,s) \neq (0,0),\ j \neq 0 \\ 2\frac{n}{\gcd(n,j)}, & \text{if } (i,s) = (0,0),\ j \neq 0 \\ 0, & \text{if } (i,s) = (0,0),\ j = 0. \end{cases}$$

$z(i,s) = \#\{1 \leq m \leq 3 \mid \ i_m \equiv 0 \mod n, \text{ where } i_1 := i, i_2 := s, i_3 := i+s\}$

Future Work

Pro-ℓ case

Set $n = \ell^k$ and study actions of B_s and $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the projective limit

$$H_1(X_H, \mathbb{Z}_\ell)$$

and the Galois representations that arise!

Generalized Heisenberg curves

As a cover of the generalized Fermat curve with Galois group $(\mathbb{Z}/n\mathbb{Z})^{s-1}$ over $\mathbb{P}^1 \setminus \{x_1, x_2, \ldots, x_{s-3}, 0, 1, \infty\}$.

Thank you for your attention!

$$\downarrow$$

$$x^n + y^n = 1$$

$$\downarrow$$

$$\mathbb{P}^1 \setminus \{0, 1, \infty\}$$