A new obstruction to the local lifting problem

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Lifting the local action

G is a finite group, k algebraically closed field of characteristic $p > 0$, $W(k)$ the ring of Witt vectors of k.

$\rho : G \hookrightarrow \mathrm{Aut}(k[[t]])$ Local G-action

Question Does there exist an extension $\Lambda/W(k)$, and a representation

$\tilde{\rho} : G \hookrightarrow \mathrm{Aut}(\Lambda[[T]])$,

such that if t is the reduction of T, then the action of G on $\Lambda[[T]]$ reduces to the action of G on $k[[t]]$?

If the answer to the above question is affirmative, then we say that the G-action lifts to characteristic zero. A group G for which every local G-action on $k[[t]]$ lifts to characteristic zero is called *a local Oort group* for k.

Know Obstructions

Some of the known obstructions to the lifting problem are:

- Bertin obstruction
- KGB obstruction,
- Hurwitz tree obstruction

The potential candidates for local Oort groups are:

- Cyclic groups
- Dihedral groups D_{p^h} of order $2p^h$
- The alternating group A_4

Results

Oort conjecture every cyclic group C_q of order $q = p^h$ is a local Oort group.

- F. Pop Ann. of Math 2014
- A. Obus and S. Wewers Ann. of Math. 2014.

- A_4 is a local Oort group A. Obus Algebra Number Theory 2016 (also known to Pop Bouw and Wewers)
- D_p are local Oort group (Bouw, Wewers Duke Math J 2006, Pagot Phd Thesis Bordeaux 2002)
- Several cases of D_{p^h} are known: A. Obus, H. Dang, S. Das, K. Karagiannis, A. Obus, V. Thatte, D_4 (B. Weaver).

Generalized Oort Conjecture

Perhaps the most significant of the currently known obstructions is the KGB obstruction (Chinburg, Guralnick, Harbater Ann. Sci. E´c. Norm. Sup. 2011).

Conjecture: If the p-Sylow subgroup of G is cyclic then this is the sole obstruction for the local lifting problem (A. Obus).

In particular, the KGB obstruction for the dihedral group D_q is known to vanish, and the so called “generalized Oort conjecture’ ’ asserts that the local action of D_q always lifts for q-odd.

Aim of the talk: Provide an “iff” criterion for $G = C_q \rtimes C_m$-action and in particular for the group D_q to lift.

Counterexample The D_{125} with a selection of lower jumps $9, 189, 4689$ does **not** lift.

Harbater-Katz-Gabber covers

A Harbater-Katz-Gabber cover (HKG) is a Galois cover $X \to \mathbb{P}^1$, such that there are at most two branched k-rational points $P_1, P_2 \in \mathbb{P}^1$, where P_1 is tamely ramified and P_2 is totally and wildly ramified.

Any finite subgroup G of $\mathrm{Aut}(k[[t]])$ can be associated with an HKG-curve X. This means that there is a HKG-curve X such that the action of $G = G(P_2)$ on $\mathcal{O}_{X,P_2} = k[[t]]$ is the original action of $G \subset \mathrm{Aut}(k[[t]])$.

Canonical embedding

X non-singular non-hyperelliptic curve, of genus $g \geq 3$ defined over an algebraically closed field with sheaf of differentials Ω_X there is the following short exact sequence:

$$0 \to I_X \to \mathrm{Sym} H^0(X, \Omega_X) \to \bigoplus_{n=0}^{\infty} H^0(X, \Omega_X^{\otimes n}) \to 0$$

where I_X is generated by elements of degree 2 and 3. Also if X is not a non-singular quintic of genus 6 or X is not a trigonal curve, then I_X is generated by elements of degree 2.

The ideal I_X is called *the canonical ideal* and it is the homogeneous ideal of the embedded curve $X \to \mathbb{P}^{g-1}$.

Canonical embedding and automorphisms

Suppose that the canonical ideal is generated by quadratic polynomials $A_1, \dots, A_r$.

A quadratic polynomial $\tilde{A}_i$ can be encoded in terms of a symmetric $g \times g$ matrix $A_i = (a_{\nu,\mu})$ as follows. Set $\bar{\omega} = (\omega_1, \dots, \omega_g)^t$. We have

$\tilde{A}_i(\bar{\omega}) = \bar{\omega}^t A_i \bar{\omega}$.

$\sigma\left(\tilde{A}_i(\bar{\omega})\right) = \bar{\omega}^t \sigma^t A_i \sigma \bar{\omega}$.

$\sigma(I_X) = I_X$ if and only if $\sigma^t A_i \sigma \in \langle A_1, \dots, A_r \rangle$ for all $1 \leq i \leq r$.

This is a linear algebra computation not involving Gröbner bases.

Example: The Fermat curve

$x^n + y^n + z^n = 0$, of genus $g = \frac{(n-2)(n-1)}{2}$ and a basis of holomorphic differentials given by $x^i y^i \omega$ for $0 \leq i \leq n-3$.

Proposition (Karagiannis, K., Terezakis, Tsouknidas)

The canonical ideal of the Fermat curve F_n consists of two sets of relations

$$G_1 = \{\omega_{i_1,j_1}\omega_{i_2,j_2} - \omega_{i_3,j_3}\omega_{i_4,j_4} : i_1+i_2 = i_3+i_4, j_1+j_2 = j_3+j_4\},$$

$$G_2 = \left\{\omega_{i_1,j_1}\omega_{i_2,j_2} + \omega_{i_3,j_3}\omega_{i_4,j_4} + \omega_{i_5,j_5}\omega_{i_6,j_6} = 0 : \substack{i_1+i_2=n+a, \quad j_1+j_2=b \\ i_3+i_4=a, \quad j_3+j_4=n+b \\ i_5+i_6=a, \quad j_5+j_6=b}\right\}$$

where $0 \leq a, b$ are selected such that $0 \leq a+b \leq n-3$.

Automorphisms of Fermat curve

Automorphism group of Fermat curves, $n = 6$, $g = 10$, using magma: algebraic set described by $g^2 = 100$ variables and 756 equations.

```
> FermatCurve(6,Rationals());
> x_{7,8}*x_{10,10} - 2*x_{9,8}*x_{9,10}
+ x_{10,8}*x_{7,10},
>
>................756 equations....................
>
> >x_{7,9}*x_{10,10} - 2*x_{9,9}*x_{9,10}
+ x_{10,9}*x_{7,10}
```

Proposition

Consider the Harbater-Katz-Gabber curve corresponding to the local group action $C_q \rtimes C_m$, where $q = p^h$ that is a power of the characteristic p. If one of the following conditions holds:

- $h \geq 3$ or $h = 2, p > 3$
- $h = 1$ and the first jump i_0 in the ramification filtration for the cyclic group satisfies $i_0 \neq 1$ and $q \geq \frac{12}{i_0 - 1} + 1$,

then the curve X has canonical ideal generated by quadratic polynomials.

Notice, that the missing cases in the above lemma which satisfy the KGB obstruction, are all either cyclic, D_3 or D_9, which are all known local Oort groups.

Liftability Criterion

Let X be a curve which is is canonically embedded in $\mathbb{P}_k^{g-1}$ and the canonical ideal is generated by quadratic polynomials, and acted on by the group G.

The curve $X \to \mathrm{Spec}(k)$ can be lifted to a family

$\mathcal{X} \to \mathrm{Spec}(R)$

along with the G-action, if and only if

- the representation $\rho_k : G \to \mathrm{GL}_g(k) = \mathrm{GL}(H^0(X, \Omega_X))$ lifts to a representation $\rho_R : G \to \mathrm{GL}_g(R) = \mathrm{GL}(H^0(\mathcal{X}, \Omega_{\mathcal{X}/R}))$
- The lift of the canonical ideal is left invariant by the action of $\rho_R(G)$.

Relative version of Petri's theorem

$R = \Lambda[[x_1, \dots, x_{3g-3}]]$ universal deformation ring of curves.

$\mathcal{X} \longrightarrow \mathrm{Spec} R$ the universal curve.

We have the following diagram relating special and generic fibres (Charalambous, Karagiannnis, K. Annales inst. Fourier 2023)

$$\begin{array}{ccccc}
\mathrm{Spec}(k) \times_{\mathrm{Spec}(R)} \mathscr{X} = \mathscr{X}_0 & \hookrightarrow & \mathscr{X} & \hookleftarrow & \mathscr{X}_\eta = \mathrm{Spec}(L) \times_{\mathrm{Spec}(R)} \mathscr{X} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Spec}(k) & \hookrightarrow & \mathrm{Spec}(R) & \hookleftarrow & \mathrm{Spec}(L)
\end{array}$$

Relative version of Petri's theorem

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & I_{\mathscr{X}_\eta} & \hookrightarrow & S_L := L[\omega_1, \dots, \omega_g] & \xrightarrow{\phi_\eta} & \bigoplus_{n=0}^{\infty} H^0(\mathscr{X}_\eta, \Omega^{\otimes n}_{\mathscr{X}_\eta / L}) & \longrightarrow & 0 \\
 & & \uparrow {\scriptstyle \otimes_R L} & & \uparrow {\scriptstyle \otimes_R L} & & \uparrow {\scriptstyle \otimes_R L} & & \\
0 & \longrightarrow & I_{\mathscr{X}} & \hookrightarrow & S_R := R[W_1, \dots, W_g] & \xrightarrow{\phi} & \bigoplus_{n=0}^{\infty} H^0(\mathscr{X}, \Omega^{\otimes n}_{\mathscr{X} / R}) & \longrightarrow & 0 \\
 & & \downarrow {\scriptstyle \otimes_R R/\mathfrak{m}} & & \downarrow {\scriptstyle \otimes_R R/\mathfrak{m}} & & \downarrow {\scriptstyle \otimes_R R/\mathfrak{m}} & & \\
0 & \longrightarrow & I_{\mathscr{X}_0} & \hookrightarrow & S_k := k[w_1, \dots, w_g] & \xrightarrow{\phi_0} & \bigoplus_{n=0}^{\infty} H^0(\mathscr{X}_0, \Omega^{\otimes n}_{\mathscr{X}_0 / k}) & \longrightarrow & 0
\end{array}
$$

Theorem

Let $f_1, \ldots, f_r \in k[\omega_1, \ldots, \omega_g]$ be quadratic polynomials which generate the canonical ideal of a curve X defined over an algebraic closed field k. Any deformation $\mathcal{X}_A$ is given by quadratic polynomials $\tilde{f}_1, \ldots, \tilde{f}_r \in A[W_1, \ldots, W_g]$, which reduce to $f_1, \ldots, f_r$ modulo the maximal ideal $\mathfrak{m}_A$ of A.

The lifting problem for a representation

$$\rho : G \to \mathrm{GL}_n(k),$$

where k is a field of characteristic $p > 0$, is about finding a local ring R of characteristic 0, with maximal ideal $\mathfrak{m}_R$ such that $R/\mathfrak{m}_R = k$, so that the following diagram is commutative:

$$\begin{array}{ccc} & & \mathrm{GL}_n(R) \\ & \nearrow & \downarrow \\ G & \longrightarrow & \mathrm{GL}_n(k) \end{array}$$

Metacyclic groups

$$G = \langle \sigma, \tau | \tau^q = 1, \sigma^m = 1, \sigma\tau\sigma^{-1} = \tau^\alpha \rangle.$$

for some $\alpha \in \mathbb{N}, 1 \leq \alpha \leq p^h - 1, (\alpha, p) = 1$.

- Dihedral groups are metacyclic $\alpha = -1$.
- $\alpha^m \equiv 1 \mod q$
- Define $\operatorname{ord}_{p^i} \alpha$ to be the smallest natural number o such that $\alpha^o \equiv 1 \mod p^i$.
- If the KGB-obstruction vanishes and $\alpha \neq 1$ then $\operatorname{ord}_{p^i} \alpha = m$ for all $1 \leq i \leq h$.

Indecomposable $k[G]$-modules

k is an algebraically closed field of characteristic $p > 0$. Indecomposable represenatations of G are determined by the value of the τ

$$\tau = \mathrm{Id}_\kappa + \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ 1 & \ddots & & & \vdots \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & 1 & 0 & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

$\tau e_i = e_i + e_{i+1}$ for $1 \leq i \leq \kappa \leq q$

And $\sigma e_1 = \zeta_m^\lambda e_1$. Using the relation $\sigma\tau\sigma^{-1} = \tau^\alpha$ we see that

$$\sigma e_i = \alpha^{i-1}\zeta_m^\lambda e_i + \sum_{\nu=i+1}^{\kappa} a_\nu e_\nu.$$

Indecomposable $k[G]$-modules

Definition $V(\lambda, \kappa)$

Indecomposable $\Lambda[G]$-modules

Λ is a local PID with $\Lambda/\mathfrak{m}_\Lambda = k$.

Jordan normal form for an element τ of order q:

$f(x) = (x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_d)$ is the minimal polynomial of τ.

$$\tau = \begin{pmatrix} \lambda_1 & 0 & \cdots & \cdots & 0 \\ a_1 & \lambda_2 & \ddots & & \vdots \\ 0 & a_2 & \lambda_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_{d-1} & \lambda_d \end{pmatrix}$$

Basis of a free module

$$
\begin{aligned}
E_1 &= E, \\
a_1 E_2 &= (T - \lambda_1 \mathrm{Id}_V) E_1, \\
a_2 E_3 &= (T - \lambda_2 \mathrm{Id}_V) E_2, \\
&\dots \\
a_{d-1} E_d &= (T - \lambda_{d-1} \mathrm{Id}_V) E_{d-1}.
\end{aligned}
$$

Let V be a free $C_q \rtimes C_m$-module, which is indecomposable as a C_q-module. Assume that no element $a_1, \dots, a_{d-1}$ is zero. The value of $\sigma(E_1)$ determines $\sigma(E_i)$ for $2 \leq i \leq d$.

Indecomposable $\Lambda[G]$-modules

$$S(a_{i-1}E_i) = S(T - \lambda_{i-1}\mathrm{Id}_V)E_{i-1} = (T^a - \lambda_{i-1}\mathrm{Id}_V)S(E_{i-1}).$$

This means that one can define recursively the action of S on all elements E_i. Indeed, assume that

$$S(E_{i-1}) = \sum_{\nu=1}^{d} \gamma_{\nu,i-1}E_\nu.$$

$$\begin{aligned}(T^a - \lambda_{i-1}\mathrm{Id}_V)E_\nu &= \sum_{\mu=1}^{d} t_{\mu,\nu}^{(\alpha)}E_\mu - \lambda_{i-1}E_\nu \\ &= (\lambda_\nu^\alpha - \lambda_{i-1})E_\nu + \sum_{\mu=\nu+1}^{d} t_{\mu,\nu}^{(\alpha)}E_\mu.\end{aligned}$$

Indecomposable $\Lambda[G]$-modules

$$a_{i-1}S(E_i) = \sum_{\nu=1}^{d} \gamma_{\nu,i-1}(\lambda_\nu^\alpha - \lambda_{i-1})E_\nu + \sum_{\nu=1}^{d} \gamma_{\nu,i-1} \sum_{\mu=\nu+1}^{d} t_{\mu,\nu}^{(\alpha)} E_\mu$$
$$= \sum_{\nu=1}^{d} \tilde{\gamma}_{\nu,i} E_\nu,$$

Proposition

$$\gamma_{i,i} \equiv \zeta_m^{\epsilon} \alpha^{i-1} \mod \mathfrak{m}$$

Let $A = \{a_1, \ldots, a_{d-1}\}$. If $a_i \in \mathfrak{m}$, then $\gamma_{\mu,i} \in \mathfrak{m}_R$ for $\mu \neq i$, that is E_i is an eigenvector for the reduced action of Γ modulo $\mathfrak{m}$.

If $a_{\kappa_1}, \ldots, a_{\kappa_r}$ are the elements of the set A which are in $\mathfrak{m}$, then the reduced matrix of Γ has the form:

$$\begin{pmatrix} \Gamma_1 & 0 & \cdots & 0 \\ 0 & \Gamma_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \Gamma_r \end{pmatrix}$$

where $\Gamma_1, \Gamma_2, \ldots, \Gamma_{r+1}$ for $1 \leq \nu \leq r+1$ are $(\kappa_\nu - \kappa_{\nu-1}) \times (\kappa_\nu - \kappa_{\nu-1})$ lower triangular matrices (we set

Reduction of indecomposables mod $\mathfrak{m}$

Theorem (K., Terezakis, J. Algebra (accepted 2024))

Consider a $k[G]$-module M which is decomposed as a direct sum

$$M = V_\alpha(\epsilon_1, \kappa_1) \oplus \cdots \oplus V_\alpha(\epsilon_s, \kappa_s).$$

The module lifts to an $\Lambda[G]$-module if and only if the set $\{1, \ldots, s\}$ can be written as a disjoint union of sets I_ν, $1 \leq \nu \leq t$ so that

- $\sum_{\mu \in I_\nu} \kappa_\mu \leq q$, for all $1 \leq \nu \leq t$.
- $\sum_{\mu \in I_\nu} \kappa_\mu \equiv a \mod m$ for all $1 \leq \nu \leq t$, where $a \in \{0, 1\}$.
- For each ν, $1 \leq \nu \leq t$ there is an enumeration $\sigma : \{1, \ldots, \#I_\nu\} \to I_\nu \subset \{1, .., s\}$, such that

$$\epsilon_{\sigma(2)} = \epsilon_{\sigma(1)} \alpha^{\kappa_{\sigma(1)}}, \epsilon_{\sigma(3)} = \epsilon_{\sigma(2)} \alpha^{\kappa_{\sigma(2)}}, \ldots, \epsilon_{\sigma(s)} = \epsilon_{\sigma(s-1)} \alpha^{\kappa_{\sigma(s-1)}}.$$

Examples

Consider the group $q = 5^2, m = 4, \alpha = 7$,

$$G = C_{5^2} \rtimes C_4 = \langle \sigma, \tau | \sigma^4 = \tau^{25} = 1, \sigma\tau\sigma^{-1} = \tau^7 \rangle.$$

Observe that $\mathrm{ord}_5 7 = \mathrm{ord}_{5^2} 7 = 4$.

The module $V_\alpha(\epsilon, 25)$ is projective and is known to lift in characteristic zero. This fits well with theorem, since $4 \mid 25 - 1 = 4 \cdot 6$.

Examples

The modules $V_\alpha(\epsilon, \kappa)$ do not lift in characteristic zero if $4 \nmid \kappa$ or $4 \nmid \kappa - 1.$

Therefore only $V_\alpha(\epsilon, 1)$, $V_\alpha(\epsilon, 4)$, $V_\alpha(\epsilon, 5)$, $V_\alpha(\epsilon, 8)$, $V_\alpha(\epsilon, 9)$, $V_\alpha(\epsilon, 12)$, $V_\alpha(\epsilon, 13)$, $V_\alpha(\epsilon, 16)$, $V_\alpha(\epsilon, 17)$, $V_\alpha(\epsilon, 20)$, $V_\alpha(\epsilon, 21)$, $V_\alpha(\epsilon, 24)$, $V_\alpha(\epsilon, 25)$ lift.

Examples

The module $V_\alpha(1,2) \oplus V_\alpha(3,2)$ lift to characteristic zero, where the matrix of τ with respect to a basis E_1, E_2, E_3, E_4 is given by

$$\tau = \begin{pmatrix} \zeta_q & 0 & 0 & 0 \\ 1 & \zeta_q^2 & 0 & 0 \\ 0 & \pi & \zeta_q^3 & 0 \\ 0 & 0 & 1 & \zeta_q^4 \end{pmatrix}$$

and $S(E_1) = \zeta_m E_1$.

Examples

The module $V_\alpha(1,2) \oplus V_\alpha(1,2)$ does not lift in characteristic zero. There is no way to permute the direct summands so that the eigenvalues of a lift S of σ are given by $\zeta_m^\epsilon, \alpha\zeta_m^\epsilon, \alpha^2\zeta_m^\epsilon, \alpha^3\zeta_m^\epsilon$. Notice that $\alpha = 2 = \zeta_m$.

Examples

The module $V_\alpha(\epsilon_1, 21) \oplus V_\alpha(2^{21} \cdot \epsilon_1, 23)$ does not lift in characteristic zero. The sum $21 + 23$ is divisible by 4, $\epsilon_2 = 2^{21}\epsilon_1$ is compatible, but $21 + 23 = 44 > 25$ so the representation of τ in the supposed indecomposable module formed by their sum can not have different eigenvalues which should be 25-th roots of unity.

We will consider a HKG-cover

$$X \xrightarrow[C_q]{} \mathbb{P}^1 \xrightarrow[C_m]{} \mathbb{P}^1, \qquad X \xrightarrow{G} \mathbb{P}^1$$

of the G-action. This has a cyclic subcover $X \overset{C_q}{\longrightarrow} \mathbb{P}^1$ with Galois group C_q.

We lift using Oort's conjecture for C_q-groups to a cover $\mathcal{X} \to \mathrm{Spec}\Lambda$.

This gives rise to a lifting

$$\begin{array}{ccc} & & \mathrm{GL} H^0(\mathcal{X}, \Omega_{\mathcal{X}/\Lambda}) = \mathrm{GL}_g(\Lambda) \\ & \nearrow & \big\downarrow \ \mathrm{mod}\ \mathfrak{m}_\Lambda \\ C_q & \longrightarrow & \mathrm{GL} H^0(X, \Omega_X) = \mathrm{GL}_g(k) \end{array}$$

Modification of the original cover

The $g \times g$ symmetric matrices $A_1, \dots, A_r$ defining the quadratic canonical ideal of the curve X, define a vector subspace of the vector space V of $g \times g$ symmetric matrices. By Oort conjecture, we know that there are symmetric matrices $\tilde{A}_1, \dots, \tilde{A}_r$ with entries in a local principal ideal domain R, which reduce to the initial matrices $A_1, \dots, A_r$.

These matrices $\tilde{A}_1, \dots, \tilde{A}_r$ correspond to the lifted relative curve $\tilde{X}$. Moreover, the submodule $\tilde{V} = \langle \tilde{A}_1, \dots, \tilde{A}_r \rangle$ is left invariant under the action of a lifting $\tilde{\rho}$ of the representation $\rho : C_q \to \mathrm{GL}_g(k)$.

Modification of the original cover

Assume that there is a lifting of the linear representaion

$$\begin{array}{ccc} & & \mathrm{GL}_g(\Lambda) \\ & \overset{\tilde{\rho}}{\nearrow} & \big\downarrow {\scriptstyle \mathrm{mod}\,\mathfrak{m}_\Lambda} \\ C_q \rtimes C_m & \xrightarrow{\rho} & \mathrm{GL}_g(k) \end{array}$$

Consider the sum of the free modules

$$W = \tilde{V} + \tilde{\rho}(\sigma)\tilde{V} + \tilde{\rho}(\sigma^2)\tilde{V} + \cdots + \tilde{\rho}(\sigma^{m-1})\tilde{V} \subset R^N = \mathrm{Sym}_g(\Lambda).$$

Modification of the original cover

- W is an $\Lambda[C_q \rtimes C_m]$-module
- Modules over PID: there is a basis $E_1, \ldots, E_N$ of Λ^N such that

$$W = E_1 \oplus \cdots \oplus E_r \oplus \pi^{a_{r+1}} E_{r+1} \oplus \cdots \oplus \pi^{a_N} E_N,$$

where $E_1, \ldots, E_r$ form a basis of $\tilde{V}$, while $\pi^{a_{r+1}} E_{r+1}, \ldots, \pi^{a_N} E_N$ form a basis of the kernel W_1 of the reduction modulo $\mathfrak{m}_\Lambda$.

Let π be the $\Lambda[C_q]$-equivariant projection map

$$W = \tilde{V} \oplus_{R[C_q]-\text{modules}} W_1 \to W_1.$$

Mascke's theorem (m is invretible in Λ) construct a module $\tilde{V}'$, which is $\Lambda[C_q \rtimes C_m]$ stable and reduces to V modulo $\mathfrak{m}_\Lambda$.

$$\bar{\pi} = \frac{1}{m} \sum_{i=0}^{m-1} \tilde{\rho}(\sigma^i) \pi \tilde{\rho}(\sigma^{-i}).$$

We see that $\bar{\pi}$ is the identity on W_1 since π is the identity on W_1. Moreover $\tilde{V}' := \ker \bar{\pi}$ is both C_q and C_m invariant and reduces to V modulo $\mathfrak{m}_\Lambda$.

Some invariant computation

$$\dim H^0(X, \Omega_X)^H = \tilde{g}$$

For the case of HKG-covers and $H = C_q$ we compute $\tilde{g} = 0$ so $H^0(X, \Omega_X)$ has no identity eigenvalues and $m \mid g$.

Bleher, Chinburg, K. (J. Num. Th. 2020) Closed formula for the multiplicities of indecomposable representations in $H^0(X, \Omega_X)$.

$$V(\lambda, \kappa) = U(\lambda + a_0(\kappa - 1) \mod m, \kappa), \quad \alpha = \zeta_m^{a_0}$$

Example 1.

Consider the curve with lower jumps $1, 21, 521$ and higer jumps $1, 5, 25$, acted on by $C_{125} \rtimes C_4$. The only possible values for α are $1, 57, 68, 124$. The value $\alpha = 1$ gives rise to a cyclic group G, while the value $\alpha = 124$ has order 2 modulo 125. The values $57, 68$ have order 4 modulo 125. The cyclic group $\mathbb{F}_5^*$ is generated by the primitive root 2 of order 4. We have that $57 \equiv 2 \mod 5$, while $68 \equiv 3 \equiv 2^3 \mod 5$.

The following modules appear in the decomposition of $H^0(X, \Omega_X)$, each one with multiplicity one.

$$U_{0,5}, U_{3,11}, U_{2,17}, U_{1,23}, U_{0,29}, U_{3,35}, U_{2,41}, U_{1,47}, U_{0,53}, U_{3,59},$$
$$U_{2,65}, U_{1,71}, U_{0,77}, U_{3,83}, U_{2,89}, U_{1,95}, U_{0,101}, U_{3,107}, U_{2,113}, U_{1,119}$$

We have that $119 \equiv 3 \mod 4$ so the module $U_{1,119}$ can not be lifted by itself. Also it can't be paired with $U_{0,5}$ since $119 + 5 \equiv 4 \neq 1 \mod 4$. All other modules have dimension d such that $d + 119 > 125$. Therefore, the representation of $H^0(G, \Omega_X)$ cannot be lifted.

The case of dihedral groups, in which the KGB-obstruction is always vanishing, is more difficult to find an example that does not lift.

The HKB-curve with lower jumps $9, 9 \cdot 21 = 189, 9 \cdot 521 = 4689$ has genus 11656 and the following modules appear in its decomposition, each one appearing with multiplicity one:

Example 2

$U_{0,1}, U_{1,1}, U_{0,2}, U_{1,2}, U_{1,3}, U_{0,4}, U_{1,4}, U_{0,5}, U_{1,6}, U_{0,7}, U_{1,7}, U_{0,8}, U_{1,8}, U_{0,9},$
$U_{1,9}, U_{0,11}, U_{1,11}, U_{0,12}, U_{1,12}, U_{0,13}, U_{1,13}, U_{0,14}, U_{1,15}, U_{0,16}, U_{0,17}, U_{1,17},$
$U_{0,18}, U_{1,18}, U_{0,19}, U_{1,19}, U_{0,21}, U_{1,21}, U_{0,22}, U_{1,22}, U_{0,23}, U_{1,23}, U_{1,24}, U_{0,25},$
$U_{1,26}, U_{0,27}, U_{1,27}, U_{0,28}, U_{1,28}, U_{0,29}, U_{1,29}, U_{0,31}, U_{1,31}, U_{0,32}, U_{1,32}, U_{0,33},$
$U_{0,34}, U_{1,34}, U_{1,35}, U_{0,36}, U_{0,37}, U_{1,37}, U_{0,38}, U_{1,38}, U_{0,39}, U_{1,39}, U_{0,41}, U_{1,41},$
$U_{0,42}, U_{1,42}, U_{0,43}, U_{1,43}, U_{1,44}, U_{0,45}, U_{0,46}, U_{1,46}, U_{1,47}, U_{0,48}, U_{1,48}, U_{0,49},$
$U_{1,49}, U_{0,51}, U_{1,51}, U_{0,52}, U_{1,52}, U_{0,53}, U_{0,54}, U_{1,54}, U_{1,55}, U_{0,56}, U_{0,57}, U_{1,57},$
$U_{0,58}, U_{1,58}, U_{0,59}, U_{1,59}, U_{0,61}, U_{1,61}, U_{0,62}, U_{1,62}, U_{0,63}, U_{1,63}, U_{1,64}, U_{0,65},$
$U_{0,66}, U_{1,66}, U_{1,67}, U_{0,68}, U_{1,68}, U_{0,69}, U_{1,69}, U_{0,71}, U_{1,71}, U_{0,72}, U_{1,72}, U_{0,73},$
$U_{1,73}, U_{0,74}, U_{1,75}, U_{0,76}, U_{0,77}, U_{1,77}, U_{0,78}, U_{1,78}, U_{0,79}, U_{1,79}, U_{0,81}, U_{1,81},$
$U_{0,82}, U_{1,82}, U_{0,83}, U_{1,83}, U_{1,84}, U_{0,85}, U_{1,86}, U_{0,87}, U_{1,87}, U_{0,88}, U_{1,88}, U_{0,89},$
$U_{1,89}, U_{0,91}, U_{1,91}, U_{0,92}, U_{1,92}, U_{0,93}, U_{1,93}, U_{0,94}, U_{1,95}, U_{0,96}, U_{1,96}, U_{0,97},$
$U_{0,98}, U_{1,98}, U_{0,99}, U_{1,99}, U_{0,101}, U_{1,101}, U_{0,102}, U_{1,102}, U_{1,103}, U_{0,104}, U_{1,104},$
$U_{0,105}, U_{1,106}, U_{0,107}, U_{1,107}, U_{0,108}, U_{1,108}, U_{0,109}, U_{1,109}, U_{0,111}, U_{1,111},$
$U_{0,112}, U_{1,112}, U_{0,113}, U_{1,113}, U_{0,114}, U_{1,115}, U_{0,116}, U_{1,116}, U_{0,117}, U_{0,118},$
$U_{1,118}, U_{0,119}, U_{1,119}, U_{0,121}, U_{1,121}, U_{0,122}, U_{1,122}, U_{0,123}, U_{1,123}, U_{1,124},$

Notice that $U_{1,123}, U_{0,123}$ can be paired with $U_{1,0}, U_{1,1}$, and then for $U_{0,121}, U_{1,121}$ there is only one $U_{1,3}$ to be paired with. The lift is not possible.

Example 3

Proposition

Assume that the first lower jump equals $b_0 = 1$ and each other lower jump is given by

$$b_\ell = \frac{p^{2\ell+1} + 1}{p + 1}.$$

Then, the local action of the dihedral group D_{p^h} lifts.

Thank you

Thank you for your attention!