A COMBINATORIAL APPROACH ON COMPUTING GRADED BETTI NUMBERS

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THE PROBLEM

Let $V = H^0(L)$ and consider the Koszul type complex:

$$\wedge^{p+1}V \otimes H^0(L^{q-1}) \xrightarrow{\delta_{p+1,q-1}} \wedge^p V \otimes H^0(L^q) \xrightarrow{\delta_{p,q}} \wedge^{p-1}V \otimes H^0(L^{q+1})$$

Our aim is the computation of

$$\dim_k(\mathrm{Tor}_p(R,K)_{p+q}) = dim_k(\frac{\mathrm{Ker}\delta_{p,q}}{\mathrm{Im}\delta_{p+1,q-1}}) = \kappa_{p,q}(S_X),$$

where $L = \Omega_X$, X Fermat curve

SYZYGIES

Let $X \subset \mathbb{P}^{g-1}$ non hyperellyptic canonical curve. Then the Betti table is given by

	0	1	$\cdots$	a	$a+1$	$\cdots$	$b-1$	b	$\cdots$	$g-3$	$g-2$
0	1	–	$\cdots$	–	–	$\cdots$	–	–	$\cdots$	–	–
1	–	β_1	$\cdots$	β_a	β_{a+1}	$\cdots$	β_{g-3-a}	–	$\cdots$	–	–
2	–	–	$\cdots$	–	β_{g-3-a}	$\cdots$	β_{a+1}	β_a	$\cdots$	β_1	–
3	–	–	$\cdots$	–	–	$\cdots$	–	–	$\cdots$	–	1

with $\beta_l = \kappa_{l,1}(S_X) = \kappa_{g-2-l,2}(S_X)$

GREEN'S CONJECTURE (1984)

$\kappa_{i,j} = 0$, for all $i < Cliff(X)$ and $j \geq 2$.

LAZARSFELD'S BUNDLE

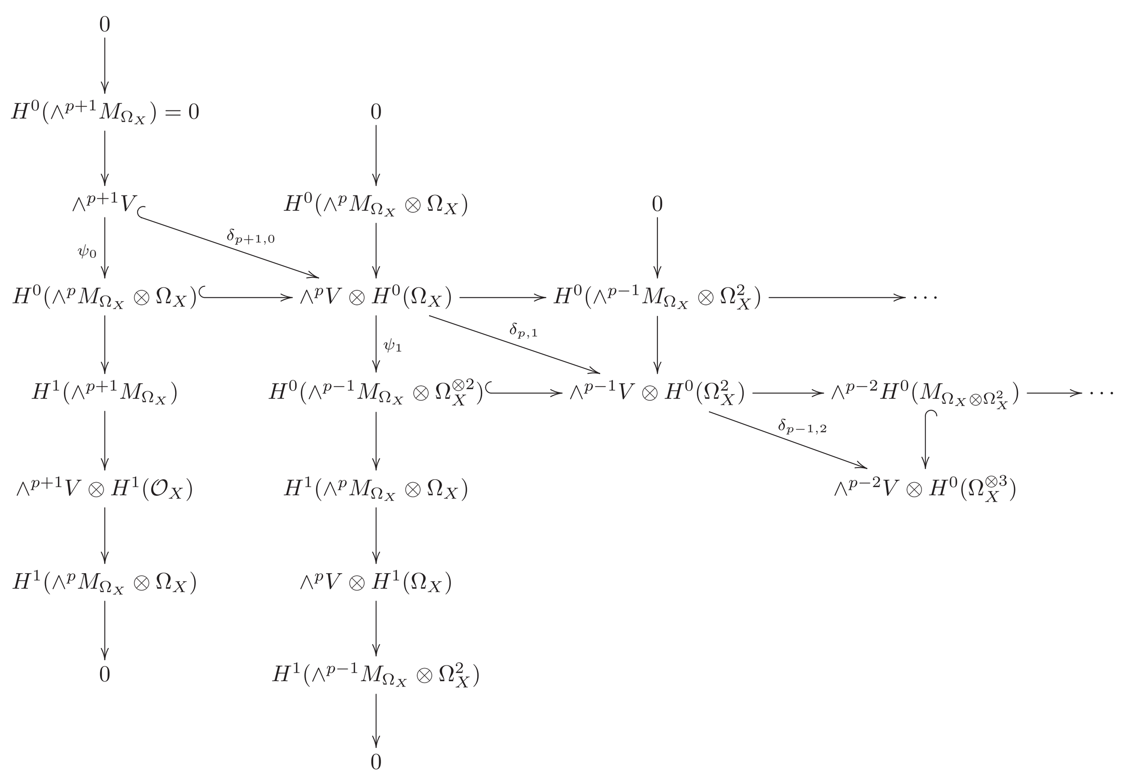
Consider the kernel bundle M_L given by the exact sequence

$$0 \to M_L \to H^0(L) \otimes_k \mathcal{O}_X \to L \to 0,$$

The wedge sequence implies

$$0 \to \wedge^p M_L \to \wedge^p (H^0(L) \otimes_k \mathcal{O}_X) \to \wedge^{p-1} M_L \otimes L \to 0$$

The interesting cases are $q = 1$ and $q = 2$. Assume that $q = 1$. Then we have the following diagram with exact columns and rows but not necessarily exact diagonal maps.



This gives us that

$$K_{p,1} = H^0(\wedge^p M_{\Omega_X} \otimes \Omega_X) / \wedge^{p+1} V.$$

Moreover

$$\mathrm{Im}\delta_{p,1} = \frac{\wedge^p V \otimes V}{\mathrm{Ker}(\delta_{p,1})} = \frac{\wedge^p V \otimes V}{H^0(\wedge^p M_{\Omega_X} \otimes \Omega_X)},$$

thus

$$K_{p-1,2} = H^0(\wedge^{p-1} M_{\Omega_X} \otimes \Omega_X^{\otimes 2}) / \mathrm{Im}(\delta_{p,1})$$

This allows us to compute

$$[\mathrm{K}_{p,1}] - [K_{p-1,2}] = -[\wedge^{p+1}V] - [H^0(\wedge^{p-1} M_{\Omega_X} \otimes \Omega_X^{\otimes 2})] + [\wedge^p V \otimes H^0(\Omega_X)]$$

FERMAT CURVE

Let $F_n : x_1^n + x_2^n + x_0^n = 0$ be a Fermat curve defined over the algebraically closed field K of characteristic $p \geq 0$. We ill work with the affine model $F_n : x^n + y^n + 1 = 0$, obtained by setting $x = \frac{x_1}{x_0}$ and $y = \frac{x_2}{x_0}$.

A K-basis for $V_m = H^0(F_n, \Omega_{F_n})$ is given by

$$\mathbf{b}_1 = \left\{ e_{ij} := \frac{x^i y^j}{y^{n-1}} dx : 0 \leq i, j, i+j \leq n-3 \right\}.$$

For every two elements e_{i_1,j_1}, e_{i_2,j_2} define the edge from vertex (i_1, j_1) to vertex (i_2, j_2)

$$E = E_{i_1,j_1,i_2,j_2} = m_t(E) e_{i_2,j_2} - m_s(E) e_{i_1,j_1},$$

connecting these two points.

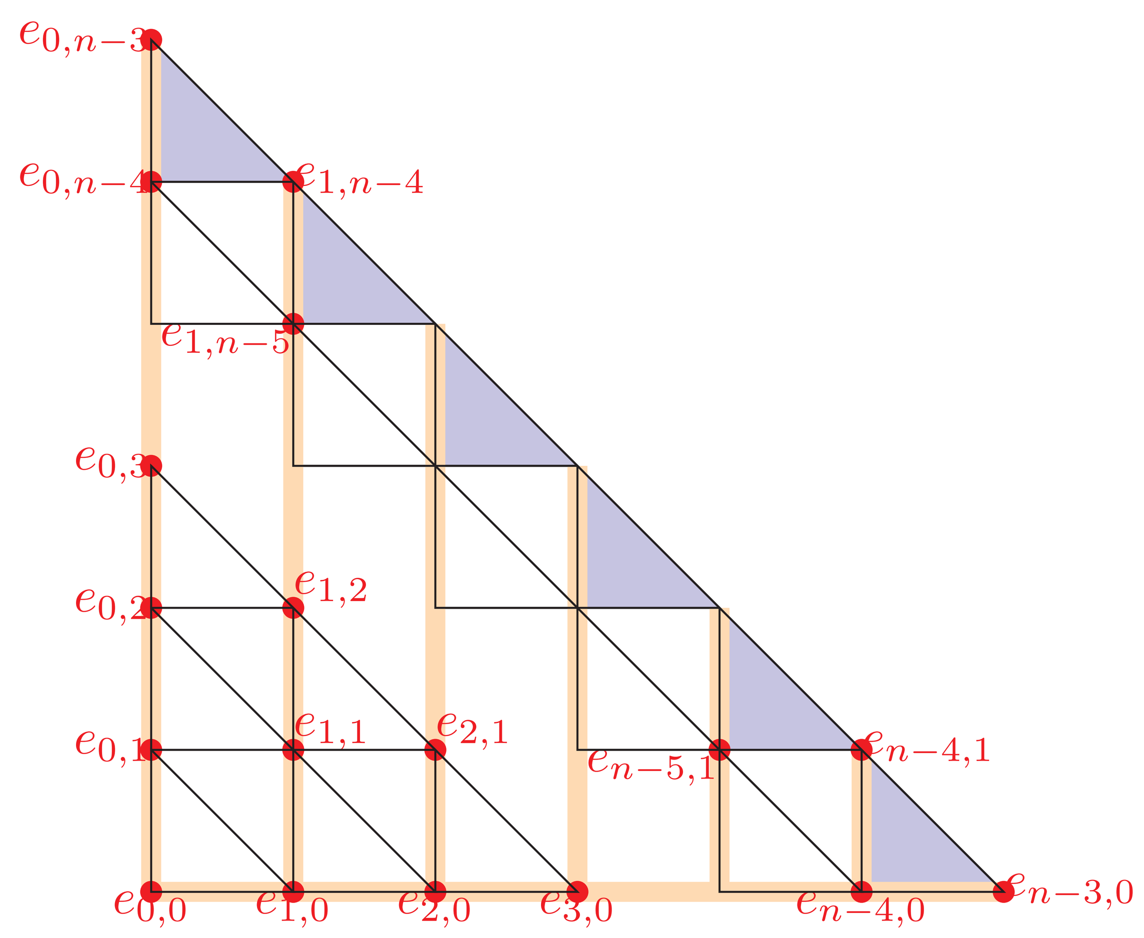
$$\frac{\mathrm{ev}(e_{i_1,j_1})}{\mathrm{ev}(e_{i_2,j_2})} = \frac{m_t(E(i_1,j_1,i_2,j_2))}{m_s(E(i_1,j_1,i_2,j_2))} = \frac{m_t(E)}{m_s(E)},$$

for two monomials $m_s(E), m_t(E) \in k[x,y]$ such that $(m_s(E), m_s(E)) = 1$.

We observe that the linear combination

$$m_t(E) e_{i_2,j_2} - m_s(E) e_{i_1,j_1} \xrightarrow{\mathrm{ev}} 0.$$

Lemma 1 *Every dependence relation corresponds to a cycle in the graph.*



RESULTS

We have calculated explicit formulas for $dim_k(H^0(\wedge^p M_{\Omega_X} \otimes \Omega_X)) = h^0(\wedge^p M_{\Omega_X} \otimes \Omega_X) = h_p$, for p=1,2,3,4 :

- $h_1 = (g-1)\binom{n-2}{2} + s(n-3) + \binom{n-4}{2}$
- $h_2 = \binom{g-1}{2}\binom{n-3}{2} + (g-1)s(n-4) + \binom{s}{2}(n-4) - \binom{s+1}{2}(n-5) + (g-1)\binom{n-5}{2} + s(n-6)$
- $h_3 = \binom{g-1}{3}\binom{n-4}{2} + \left(s\binom{g-1}{2} + \binom{s}{2}(g-1) + \binom{s}{3}\right)(n-5) - \left(\binom{s+1}{2}(g-1+s)(n-6) - \binom{s+2}{3}(n-6+n-7)\right) + \binom{g-1}{2}\binom{n-6}{2} + (g-1)s(n-7) + \binom{s}{2}(n-7) - \binom{s+1}{2}(n-8) + (g-1)\binom{n-8}{2} + s(n-9)$
- $h_4 = \binom{g-1}{4}\binom{n-5}{2} + \sum_{i=0}^{3}\binom{g-1}{i}\binom{s}{4-i}(n-6) + \binom{s+2}{3}(g-1)(n-8) - \binom{s+3}{4}(n-9) - \binom{s_2}{3}(g-1)(n-7) - (3\binom{s}{4} + 2s\binom{s-1}{2} + s(s-1) + \binom{s}{2})(n-8) + (-\binom{s+1}{2}\binom{g-1}{2} + 3\binom{s}{4} + \binom{s+1}{2}(g-1)(n-7) + s\binom{s-1}{2})(n-7)$

where $g = g(F_n) = \binom{n-1}{2}, s = \binom{n-2}{2}$

FUTURE WORK

- Give a combinatorial formula for $h^0(\wedge^p M_{\Omega_X} \otimes \Omega_X)$, for all $1 \leq p \leq g-3$
- Calculate equivariant Betti numbers for Fermat Curve

REFERENCES

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