# Actions on the Homology of the Heisenberg Curve

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## Heisenberg Curve

Let $H_n = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},\ x, y, z, \in \mathbb{Z}/n\mathbb{Z} \right\}$

which is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^2 \rtimes \mathbb{Z}/n\mathbb{Z}$. We can define a curve that is an $H_n$-cover of $\mathbb{P}^1$, ramified above three points by the topological Galois correspondence of

$$\pi_1(\mathbb{P}^1 - \{0, 1, \infty\}) = F_2 :$$

$$1 \to R_{\text{Heis}} \to F_2 \to H_n \to 1$$

Is it of any interest?

## Key Steps

1. Compute the fundamental group of the open Heisenberg curve using tools from combinatorial group theory and describe the classes of loops on the punctured $g$-holed torus.
2. Define the Galois action by conjugation on the abelianization on the $g$-torus.
3. Quotient by a proper subgroup, which tracks the ramification data of the cover, gives us the compact $g$-torus.
4. Adapt an idea from arithmetic topology to perform representation theory on the curve and set the ground for future work.

## Fermat Curve

The affine equation $x^n + y^n = 1$ is the Fermat curve, which can be seen as a ramified cover of $\mathbb{P}^1$ with Galois group $(\mathbb{Z}/n\mathbb{Z})^2$.

$$\pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\}) = \langle a, b \rangle \quad \supset \quad \langle a^n, b^n, [a,b] \rangle \quad \supset \quad \langle a^n, b^n, [a,[a,b]], [b,[a,b]] \rangle \quad \supset \quad 1$$

$$\tilde{X} \to \text{Heisenberg} \xrightarrow{\mathbb{Z}/n\mathbb{Z}} \text{Fermat} \xrightarrow{(\mathbb{Z}/n\mathbb{Z})^2} \mathbb{P}^1 - \{0,1,\infty\}, \qquad \text{Heisenberg} \xrightarrow{H_n} \mathbb{P}^1 - \{0,1,\infty\}$$

The Heisenberg curve is a cover of the Fermat curve, ramified above $\infty$ when $n$ is even and unramified otherwise.

## Schreier's Lemma

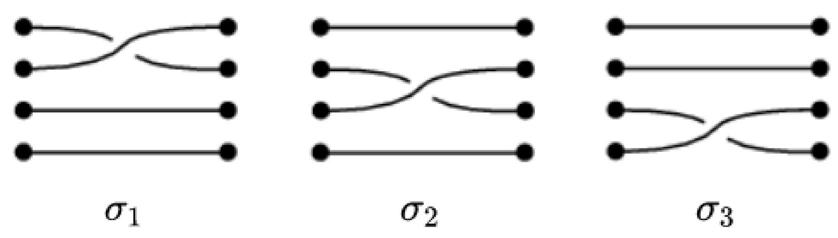
Let $F(X)$ be a free group of finite rank and $H$ be a subgroup. A Schreier transversal $T$ for $H$ is a finite complete set of coset representatives, such that for every word $t$ in $T$ every initial segment of $t$ is also in $T$. For $g$ in $F$ denote by $\overline{g}$ the unique element of $T$ such that $Hg = H\overline{g}$. Then $H$ is freely generated by

$$\{tx(\overline{tx})^{-1} \neq 1 \mid t \in T, x \in X\}$$

## Arithmetic Topology

The analogy between primes and knots is expressed in numerous ways, two of which are

1. Pure braids and $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, via Artin and Ihara representations.
   Let $F_{s-1} = \langle x_1, x_2 \ldots, x_s \mid x_1 x_2 \cdots x_s \rangle$ be the free group on $s-1$ generators. The braid group $B_{s-1}$ can be realized as a subgroup of $\text{Aut}(F_{s-1})$ generated by the elements $\sigma_i$:




$$\sigma_i(x_{i+1}) = x_i$$
$$\sigma_i(x_i) = x_i x_{i+1} x_i^{-1}$$
$$\sigma_i(x_k) = x_k,\ k \neq i, i+1$$

   There is a natural surjection to the symmetric group $B_{s-1} \longrightarrow S_{s-1}$ and pure braids are the kernel of this map. In particular, for a pure braid $\sigma$ we have that $\sigma(x_i) \sim x_i^n$, where $\sim$ denotes conjugation and $n \in \mathbb{N}$.

   The Ihara representation $\text{Ih}_s : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \text{Aut}(\mathfrak{F}_{s-1})$ where $\mathfrak{F}_{s-1}$ is the pro-$\ell$ completion of $F_{s-1}$ has image inside the subgroup

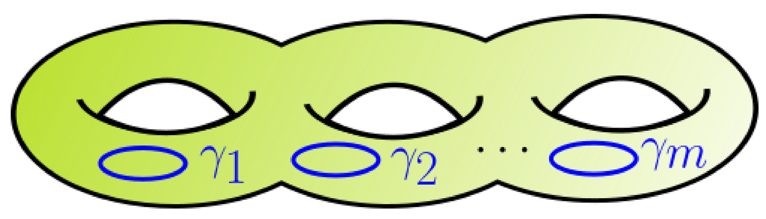
$$\{\sigma \in \text{Aut}(\mathfrak{F}_{s-1}) \mid \sigma(x_i) \sim x_i^{N(\sigma)},\ N(\sigma) \in \mathbb{Z}_\ell^\times\}$$

2. The Link group $\pi_1(M \setminus K_1 \cup \cdots \cup K_r)$ and the Galois group with restricted ramification $\pi_1(\text{Spec}(\mathcal{O}_k) \setminus \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\})$, via Alexander modules.

   Let $G = \langle x_1, \ldots, x_s \mid R_1 = \cdots = R_r = 1 \rangle$ be one of the two groups, pro-finite in the second case, and $\psi : G \longrightarrow H$ an epimorphism. Let $N := \ker\psi$ and $\mathcal{A}_\psi$ the (pro-$\ell$) Alexander module corresponding to $\psi$. The Link module $N^{\text{ab}}$ (resp. Iwasawa module) is understood through the Crowell exact sequence of $\mathbb{Z}[H]$- (resp. $\mathbb{Z}_\ell[[H]]$-) modules:

$$1 \to N^{\text{ab}} \to \mathcal{A}_\psi \to \mathbb{Z}[H] \to \mathbb{Z} \to 1, \quad 1 \to N^{\text{ab}} \to \mathcal{A}_\psi \to \mathbb{Z}_\ell[[H]] \to \mathbb{Z}_\ell \to 1$$

## Compactified Curve




The fundamental group $R_{\text{Heis}}$ admits a presentation, $g$ being the genus of the curve:

$$\langle a_1, b_1, \ldots, a_g, b_g, \gamma_1, \ldots, \gamma_m \mid [a_1, b_1] \cdots [a_g, b_g] \gamma_1 \cdots \gamma_m = 1 \rangle$$

Denote by $X_H$ the closed curve, after annihilating the elements $\gamma_i$ by quotienting with $\Gamma = \langle a^n, b^n, (ab)^n \rangle$. We apply the Crowell exact sequence on the homology

$$H_1(X_H, \mathbb{Z}) = \frac{R^{\text{ab}}_{\text{Heis}} \cdot \Gamma}{\Gamma}$$

## Irreducible Characters

The irreducible characters of $H_n = \langle \alpha, [\alpha, \beta] \rangle \rtimes \langle \beta \rangle$ can be computed using the general method for semi-direct products when the normal group is abelian.

$$\chi_{ijs}(g) = \sum_{h \in cl(g)} (\chi_{ij} \otimes \chi_s)(h)$$

with $\zeta = e^{\frac{2i\pi}{n}}$, $\alpha \mapsto \zeta^i$, $[\alpha, \beta] \mapsto \zeta^j$, $\beta \mapsto \zeta^s$.

## References

[1] J. Antoniades and A. Kontogeorgis. The group of automorphisms of the Heisenberg curve. *Abelian Varieties and number theory*, Vol. 767 of Contemp. Math., pages 25-39. Amer. Math. Soc., 2021.

[2] A. Kontogeorgis and P. Paramantzoglou. Galois action on the homology of generalized Fermat curves. *Q. J. Math.*, 71(4):1377-1417, 2020.

[3] M. Morishita. *Knots and Primes: An Introduction to Arithmetic Topology*. SpringerLink: Bücher. Springer-Verlag London Limited, 2011.

## Result

Let $\mathbb{F}$ be a field containing the $n$-th roots of unity. We have described the homology of the closed Heisenberg curve as an $\mathbb{F}[H_n]$-module:

$$H_1(X_H, \mathbb{F}) = \bigoplus_{i,j=0}^{n-1} \bigoplus_{s=0}^{\gcd(n,j)-1} \mathbb{F}h_{ijs}\chi_{ijs},$$

where

$$h_{ijs} = \begin{cases} 1 - z(i,s) & \text{if } (i,s) \neq (0,0),\ j = 0 \\ \frac{n}{\gcd(n,j)}, & \text{if } (i,s) \neq (0,0),\ j \neq 0 \\ 2\frac{n}{\gcd(n,j)}, & \text{if } (i,s) = (0,0),\ j \neq 0 \\ 0, & \text{if } (i,s) = (0,0),\ j = 0. \end{cases}$$

$$z(i,s) = \#\{1 \leq m \leq 3 \mid i_m \equiv 0 \mod n,$$
$$\text{where } i_1 := i, i_2 := s, i_3 := i + s\}$$

## Future Work

1. Make use of Ihara's vision. Let $G$ be $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ or the braid group $B_s$. Compute the action of $G$ on $H_1(X_H, \mathbb{Z}_\ell)$, specifically on the basis generators and study the Galois Representations that arise.
2. Find a suitable generalization of the Heisenberg curve as a cover of the punctured projective line $\mathbb{P}^1 - \{x_1, x_2, \ldots, x_{s-3}, 0, 1, \infty\}$ extended by the group $(\mathbb{Z}/n\mathbb{Z})^{s-1}$ for all $s$ greater than 3.