TRANSFER & NORM FOR FINITE GROUP SCHEMES

Kostas Karagiannis,
joint work with Peter Symonds

National and Kapodistrian University of Athens

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Motivation

Trace & Norm in pure mathematics

Abstract

Traces and determinants are fundamental tools of linear algebra with ubiquitous influence throughout several branches of modern algebra...

List a few occurrences of traces/transfers and norms in pure mathematics.

Trace & Norm in field theory

Let L/K be an an extension of number fields. For each $\alpha \in L$ let

$$m_\alpha : L \to L,\ \beta \mapsto \alpha\beta,$$

and let M_α be the matrix of m_α with respect to some K-basis of L.

Definition

$$\mathrm{Tr} : L \to K,\ \alpha \mapsto \mathrm{Trace}(M_\alpha), \quad \mathrm{Nm} : L \to L,\ \alpha \mapsto \mathrm{Det}(M_\alpha).$$

Applications: Rings of integers, discriminants, ramification theory, class field theory, Galois/Tate cohomology and many more.

Equivalently, if $\sigma_1, \ldots, \sigma_n$ are the K-embeddings of L into $\mathbb{C}$, then

$$\mathrm{Tr}(\alpha) = \sum_{i=1}^{n} \sigma_i(\alpha), \quad \mathrm{Nm}(\alpha) = \prod_{i=1}^{n} \sigma_i(\alpha).$$

This definition extends to finite separable field extensions, and to finite free commutative ring extensions $R \to S$.

Trace & Norm for finite groups

Let k be a field, G a finite group, H a subgroup, M a kG-module.

Definition

Define $\mathrm{Tr} : M^H \to M^G$, $\mathrm{Nm} : M^H \to M^G$ as

$$\mathrm{Tr}(m) = \sum_{\overline{g} \in [G/H]} \overline{g} m, \quad \mathrm{Nm}(m) = \prod_{\overline{g} \in [G/H]} \overline{g} m.$$

This can be further extended to group cohomology, giving maps

$$\mathrm{Tr} : H^n(H, M) \to H^n(G, M), \quad \mathrm{Nm} : H^n(H, M) \to H^n(G, M).$$

Applications: Detection of (relative) projectivity, invariant theory, spectral sequences, finite generation of the cohomology ring.

Generalizations?

Generalize the object acted upon

1. Purely inseparable field extensions.
2. Commutative rings with trivial automorphism groups.
3. Non-Galois covers of schemes.

Generalize the acting object

1. Actions of Lie algebras.
2. Modules for arbitrary finite dimensional algebras.
3. Hochschild cohomology.

Generalize the base object

1. Actions on modules over a general commutative ring.
2. Equivariant quasicoherent sheaves on schemes.

Finite group schemes

Finite group schemes

A finite group scheme G over k can be defined as:

Definition

1. A group object in the category of affine k-schemes.
2. The prime spectrum of a finite dimensional commutative Hopf algebra.
3. The prime spectrum of the k-dual of a finite dimensional cocommutative Hopf algebra.
4. The finite scheme that represents a group-valued functor.

Group objects

Let $\mathcal{C}$ be a locally small category with finite products.[1]

Definition

A group object G of $\mathcal{C}$ is one that comes equipped with morphisms

$$m : G \times G \to G, \; e : * \to G \text{ and } i : G \to G$$

satisfying the usual group axioms (associativity, unitality, inverse).

Examples:

- The group objects of (Sets) are groups.
- The group objects in (Smooth Manifolds) are Lie groups.
- The group objects in (AffSch) are finite group schemes.

[1] Sets, Grps, Top, CRngs, AffSch...

Hopf algebras

Definition

A Hopf algebra is a k-algebra H equipped with homomorphisms

$$\Delta : H \to H \otimes H,\ \epsilon : H \to k,\ S : H \to H$$

called comultiplication, counit and antipode repectively, satisfying

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta, \quad \text{(coassociativity)}$$
$$(\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \epsilon) \circ \Delta, \quad \text{(counitality)}$$

Spec(Hopf algebra)=group object

Let H be a Hopf algebra with comult. $\Delta : H \to H \otimes H$, satisfying

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta.$$

Applying the Spec functor and setting $G = \mathrm{Spec}(H)$

$$m = \mathrm{Spec}(\Delta) : G \times G \to G, \; m \circ (m \times \mathrm{id}) = m \circ (\mathrm{id} \times m)$$

Applying vector space duality and setting $H^* = \mathrm{Hom}_k(H, k)$

$$\mu = m^* : H^* \otimes H^* \to H^*, \; \mu \circ (\mu \otimes \mathrm{id}) = \mu \circ (\mathrm{id} \otimes \mu)$$

One may recover any of $H, H^*, \mathrm{Spec}(H)$ from any of the other two.

Conclusion

G is a group object $\Leftrightarrow k[G]$ is Hopf $\Leftrightarrow kG = k[G]^*$ is Hopf.

Modules

It is more intuitive (?) to think of G-modules as modules over the finite dimensional k-algebra kG.

Properties: Frobenius, cocommutative, Hopf, self-injective, tensor category, generally wild representation type.

Non-properties: Symmetric, commutative, hereditary.

Standard examples: Group algebra of a finite group, universal enveloping algebra of a finite dimensional p-restricted Lie algebra.

If H is a closed subgroup scheme of G then kH is a subalgebra of kG, and the latter is free over the former.

Trace/Transfer

Restriction, Induction, Coinduction

Let H be a closed subgroup scheme of a finite group scheme G over a field k. The inclusion $f : kH \hookrightarrow kG$ gives rise to functors

$$(H\text{-modules}) \; \underset{f_* = \mathrm{Coind}_H^G \;\rightarrow}{\overset{f_! = \mathrm{Ind}_H^G \;\rightarrow}{\xleftarrow[f^* = \mathrm{Res}_H^G]{}}} \; (G\text{-modules})$$

- $\mathrm{Res}_H^G(-)$ is the restriction of scalars.
- $\mathrm{Ind}_H^G(-) = - \otimes_{kH} kG$ is the extension of scalars or induction.
- $\mathrm{Coind}_H^G(-) = \mathrm{Hom}_H(kG, -)$ is the coinduction.

Units and Counits

By Frobenius reciprocity, these functors form an adjoint triple:

$$f_! \dashv f^* \dashv f_* \text{ or equivalently } \mathrm{Ind}_H^G \dashv \mathrm{Res}_H^G \dashv \mathrm{Coind}_H^G.$$

Each adjunction is uniquely determined a pair of natural transformations (η, ϵ) called the unit/counit pair of the adjunction.

If M is a G-module and N is an H-module,

$$\eta_L : \mathbf{1}_{H\text{-mod}} \to \mathrm{Res}_H^G \mathrm{Ind}_H^G, \qquad \epsilon_L : \mathrm{Ind}_H^G \mathrm{Res}_H^G \to \mathbf{1}_{G\text{-mod}}$$
$$\eta_R : \mathbf{1}_{G\text{-mod}} \to \mathrm{Coind}_H^G \mathrm{Res}_H^G, \quad \epsilon_R : \mathrm{Res}_H^G \mathrm{Coind}_H^G \to \mathbf{1}_{H\text{-mod}}.$$

The case of finite groups

Let H be a subgroup of a finite group G, and let $f : kH \hookrightarrow kG$.

Theorem (The Nakayama isomorphism)

There exists a natural isomorphism $\mathrm{Coind}_H^G(-) \cong \mathrm{Ind}_H^G(-)$,

$$\mathrm{Hom}_{kH}(kG, M) \quad \to \quad M \otimes_{kH} kG,\ \phi \mapsto \sum_{\overline{g} \in [G/H]} \phi(\overline{g}) \otimes \overline{g}^{-1}.$$

Thus induction and restriction are biadjoint, $\mathrm{Ind}_H^G \dashv \mathrm{Res}_H^G \dashv \mathrm{Ind}_H^G$.

This is controlled by the invariants of the left regular representation

$$(kG)^G = k \cdot \sum_{g \in G} g = k.$$

Transfer for finite groups revisited

The unit η_R of $\mathrm{Res}_H^G \dashv \mathrm{Ind}_H^G$ and the counit ϵ_L of $\mathrm{Ind}_H^G \dashv \mathrm{Res}_H^G$ are

$$\begin{aligned} \epsilon_L(M) &: kG \otimes_{kH} M \to M, \quad g \otimes_{kH} m \mapsto gm, \\ \eta_R(M) &: M \to kG \otimes_{kH} M, \quad m \mapsto \sum_{\overline{g} \in [G/H]} \overline{g} \otimes \overline{g}^{-1} m, \end{aligned}$$

Taking $m \in M^H = \mathrm{Hom}_H(k, M)$ one can check that

$$k \xrightarrow{\eta_R(M)} kG \otimes_{kH} k \xrightarrow{\mathrm{Ind}_H^G(m)} kG \otimes_{kH} M \xrightarrow{\epsilon_L(M)} M$$

defines an element of $\mathrm{Hom}_G(k, M) = M^G$ given by

$$\sum_{\overline{g} \in [G/H]} \overline{g} m = \mathrm{Tr}(m).$$

The Wirthmüller isomorphism

Let G be a finite group scheme, H a closed subgroup scheme.

Definition

The *modular function* δ_G of G is the one-dimensional G-module $(kG)^G$. One says that G is *unimodular* if δ_G is trivial.

Theorem (Wirthmüller isomorphism)

There exists a natural isomorphism of functors

$$W_{H,G} : \mathrm{Coind}_H^G(-) \xrightarrow{\cong} \mathrm{Ind}_H^G(- \otimes \mu_{H,G}^{-1})$$

where $\mu_{H,G}$ is the H-module $\mathbf{R}_H^G(\delta_G) \otimes (\delta_H)^* = \mathbf{R}_H^G(\delta_G)(\delta_H)^{-1}$.

Thus $\mathrm{Ind}_H^G \dashv \mathrm{Res}_H^G \dashv \mathrm{Ind}_H^G(- \otimes \mu_{H,G}^{-1})$, and

$\eta_R : \mathbf{1}_{G\text{-mod}} \to \mathrm{Ind}_H^G(\mathrm{Res}_H^G(- \otimes \mu_{H,G}^{-1}))$, $\epsilon_L : \mathrm{Ind}_H^G \mathrm{Res}_H^G \to \mathbf{1}_{G\text{-mod}}$.

Transfer for finite group schemes

Definition (K.-Symonds 2023)

Let M be a G-module. The *relative transfer* from H to G is

$$\mathrm{Tr}_H^G : (\mu_{H,G} \otimes M)^H \to M^G, \quad f \mapsto \epsilon_L(M) \circ \mathrm{Ind}_H^G(f) \circ \eta_R(k).$$

That is, any $f \in (\mu_{H,G} \otimes M)^H = \mathrm{Hom}_H(\mu_{H,G}^{-1}, M)$ gives rise to an element of $\mathrm{Hom}_G(k, M) = M^G$ via

$$\begin{array}{ccc} k & \xrightarrow{\mathrm{Tr}_H^G(f)} & M \\ \downarrow{\scriptstyle \eta_R(k)} & & \uparrow{\scriptstyle \epsilon_L(M)} \\ \mathrm{Ind}_H^G(\mu_{H,G}^{-1}) & \xrightarrow{\mathrm{Ind}_H^G(f)} & \mathrm{Ind}_H^G(M). \end{array}$$

Transitivity, Base change, Functoriality

Let M, M' be G-modules. The above can be generalized to define

$$Tr_H^G : \mathrm{Hom}_H(M' \otimes \mu_{H,G}^{-1}, M) \to \mathrm{Hom}_G(M', M),$$
$$\mathrm{Tr}_H^G : \mathrm{Ext}_H(M' \otimes \mu_{H,G}^{-1}, M) \to \mathrm{Ext}_G(M', M),$$

a special case of which is a transfer for group scheme cohomology.

Proposition (K.Symonds 2024)

Let $K \leq H \leq G$, L/k a field extension, and f, g, h Ext-classes.

1. $\mathrm{Tr}_H^G \circ \mathrm{Tr}_K^H = \mathrm{Tr}_K^G$.
2. $\mathrm{Tr}_H^G \otimes L = \mathrm{Tr}_{H_L}^{G_L}$.
3. $h \circ \mathrm{Tr}_H^G(f) \circ g = \mathrm{Tr}_H^G(h \circ f \circ (g \otimes \mathbf{1}_{\mu_{H,G}^{-1}}))$.

Transfer detects (relative) projectivity

Proposition (K.-Symonds 2024)

If G is unipotent then $\mathrm{Tr}_H^G \circ \mathrm{res}_H^G = 0$.

Proposition (Higman's criterion, K.-Symonds 2023)

The following conditions on a G-module M are equivalent.

1. *M is projective relative to a closed subgroup scheme H.*
2. *M is isomorphic to a direct summand of $\mathrm{Ind}_H^G(M)$.*
3. *id_M is in the image of the transfer map Tr_H^G.*

Corollary (K.-Symonds 2023)

If $\mathrm{char}(k) \nmid |G : H|$, every G-module is projective relative to H.

Mackey Theory

Recall that if $H, K \leq G$ are finite groups and M is a G-module then one has the following *Mackey's double coset formulas*

$$\mathrm{Res}^G_H \mathrm{Ind}^G_K M \cong \bigoplus_{g \in [H \backslash G / K]} \mathrm{Ind}^H_{H \cap {}^g K} \mathrm{Res}^{{}^g K}_{H \cap {}^g K} ({}^g M)$$

$$\mathrm{Res}^G_H \circ \mathrm{Tr}^G_K = \sum_{g \in [H \backslash G / K]} \mathrm{Tr}^H_{H \cap {}^g K} \circ c_g^* \circ \mathrm{Res}^K_{H^g \cap K}$$

Theorem (Feshbach 1996, K.-Symonds 2024)

Mackey's double coset formula holds if $K, H \leq G$ are unimodular finite group schemes over an algebraically closed field.

Norm

Infinitesimal group schemes

Let k be a field of characteristic $p > 3$. If L/k is a purely inseparable extension then L has no non-trival k-automorphisms.

A finite group scheme G over k is called infinitesimal if its coordinate ring $k[G]$ is a local ring. Over a perfect field

$$G = \mathrm{Spec}\left(k[X_1, \ldots, X_n]/(X_1^{p^{a_1}}, \ldots, X_n^{p^{a_n}})\right)$$

Theorem (Childs, 1976)

For any purely inseparable finite extension L/k, there exists an infinitesimal group scheme G acting on L such that $L^G = k$.

Intuitively, infinitesimal group schemes are as far away as possible from finite groups, and close as possible to Lie algebras.

Étale group schemes

At the other end one has so-called étale (finite) group schemes whose coordinate rings are (finite) étale/separable k-algebras,

$$\operatorname{Spec}(L_1 \times \cdots \times L_n), \text{ with each } L_i/k \text{ finite separable.}$$

If all $L_i = k$, one recovers the coordinate ring of a finite group.

The base change $G^{\text{ét}}_{k^{\text{sep}}}$ of a finite étale group scheme to a separable closure k^{sep} of k can be indentified with a finite group.

Observation (K.-Symonds, 2024)

1. If G is a finite étale group scheme acting on a field L, then L/L^G is finite separable.
2. If L/K is a finite Galois extension, then $G_{k^{\text{sep}}} = \mathrm{Gal}(L/K)$.

Cartier's structure theorem

As every finite extension can be built from a separable and a purely inseparable field extension, one has that every finite group scheme G can be built from an étale and an infinitesimal group scheme.

Every finite group scheme G has a largest étale quotient, denoted $G^{\text{ét}}$. The kernel G^0 of the projection map $G \twoheadrightarrow G^{\text{ét}}$ is an infinitesimal group scheme.

Theorem (Cartier's Theorem)

Every finite group scheme G over a perfect field k can be written as a semi-direct product $G = G^0 \rtimes G^{\acute{e}t}$.

The definition of the norm

A G-algebra S is a commutative k-algebra equipped with a G-module structure compatible with its ring structure.

Definition (K.-Symonds 2024)

Let S be a G-algebra. The *relative norm* from H to G is

$$\mathrm{Nm}_H^G : S^H \to S^G,\ s \mapsto \prod_{g \in [G^{\text{ét}}/H^{\text{ét}}]} gs^{|G^0:H^0|}.$$

Properties of the norm

Proposition (K.-Symonds 2024)

1. $\mathrm{Nm}_H^G(s) = s^{|G:H|}$ *for all* $s \in S^G$.
2. $\mathrm{Nm}_H^G \circ \mathrm{Nm}_K^H = \mathrm{Nm}_K^G$ *for any* $K \leq H \leq G$.
3. $\mathrm{Nm}_H^G(rs) = \mathrm{Nm}_H^G(r)\mathrm{Nm}_H^G(s)$, *for all* $r, s \in S^H$.
4. $\mathrm{Nm}_H^G(f(s)) = f\left(\mathrm{Nm}_H^G(s)\right)$ *for all* $s \in S^H$ *and all* G*-algebra homomorphisms* $f : S \to R$.

Theorem (K.-Symonds 2024)

Let L *be a field which is a* G*-algebra. Then* $[L : L^G] \mid |G|$ *and*

$$\mathrm{Nm}_{\{1\}}^G(\alpha) = \mathrm{N}_{L/L^G}(\alpha)^{\frac{|G|}{[L:L^G]}}, \ \forall \, \alpha \in L$$

where $N_{L/L^G} : L \to L^G$ *is the classic field norm.*

Thank you for your attention!

What should I write in the last slide of a pure math beamer presentation to thank the audience in a witty manner?