Gorenstein modules and dimension over large families of infinite groups

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The talk is based on the paper:

Stergiopoulou, D.-D.: Gorenstein modules and dimension over large families of infinite groups. Collectanea Mathematica (to appear)

Contents:

- Overview and a central question
- Basic definitions
- Theorems
- Some usefull tools
- References

Gorenstein homological algebra is the relative homological algebra based on Gorenstein projective, Gorenstein injective and Gorenstein flat modules.

In classical homological algebra every projective module is flat.

Question: Is the class of Gorenstein projective modules contained in the class of Gorenstein flat modules?

In this talk, we consider the case where the base ring is a group ring.

Gorenstein modules: Let R be a ring.

A module M is Gorenstein projective if it is a syzygy of an acyclic complex

$$\mathcal{P} = \cdots \to P_{i+1} \to P_i \to P_{i-1} \to \cdots$$

of projectives such that the complex $\mathrm{Hom}_R(\mathcal{P}, Q)$ is acyclic for every projective module Q.

A module M is Gorenstein flat if it is a syzygy of an acyclic complex

$$\mathcal{F} = \cdots \to F_{i+1} \to F_i \to F_{i-1} \to \cdots$$

of flats such that the complex $I \otimes_R \mathcal{F}$ is acyclic for every injective module I.

A module M is Gorenstein injective if it is a syzygy of an acyclic complex

$$\mathcal{I} = \cdots \to I_{i+1} \to I_i \to I_{i-1} \to \cdots$$

of injectives such that the complex $\mathrm{Hom}_R(J, \mathcal{I})$ is acyclic for every injective module J.

A module M is projectively coresolved Gorenstein flat if it is a syzygy of an acyclic complex

$$\mathcal{P} = \cdots \to P_{i+1} \to P_i \to P_{i-1} \to \cdots$$

of projectives such that the complex $I \otimes_R \mathcal{P}$ is acyclic for every injective module I.

Projectively coresolved Gorenstein flat modules (PGF) were introduced by Saroch and Stovicek (2020)

They showed that every PGF module is Gorenstein projective.

It is clear that every PGF module is Gorenstein flat.

$$\begin{tikzcd}
\mathrm{GProj}(R) & & \mathrm{GFlat}(R) \\
& \mathrm{PGF}(R) \arrow[ul, hook, bend left] \arrow[ur, hook, bend right] &
\end{tikzcd}$$

Question: Is the class of Gorenstein projective modules contained in the class of Gorenstein flat modules?

$\Updownarrow$

Question: Is the class of Gorenstein projective modules equal to the class of projectively coresolved Gorenstein flat modules?

The questions are equivalent!

A module M is weak Gorenstein projective if it is a syzygy of an acyclic complex

$$\mathcal{P} = \cdots \longrightarrow P_{i+1} \longrightarrow P_i \longrightarrow P_{i-1} \longrightarrow \cdots$$

of projectives.

A module M is weak Gorenstein injective if it is a syzygy of an acyclic complex

$$\mathcal{I} = \cdots \longrightarrow I_{i+1} \longrightarrow I_i \longrightarrow I_{i-1} \longrightarrow \cdots$$

of injectives.

A module M is weak Gorenstein flat if it is a syzygy of an acyclic complex

$$\mathcal{F} = \cdots \longrightarrow F_{i+1} \longrightarrow F_i \longrightarrow F_{i-1} \longrightarrow \cdots$$

of flats.

$$
\begin{tikzcd}[column sep=small]
\mathrm{PGF}(R) & & & & & & & & \mathrm{GFlat}(R) \\
& ?\ \mathcal{A} \arrow[ul, hook] & & & & & & \mathcal{B}\ ? \arrow[ur, hook] & \\
& & \mathrm{WGProj}(R) \arrow[ul, hook] & & & & \mathrm{WGFlat}(R) \arrow[ur, hook] & & \\
& & & \mathrm{GProj}(R) \arrow[ul, hook] & & \mathrm{GFlat}(R) \arrow[ur, hook] & & & \\
& & & & \mathrm{PGF}(R) \arrow[ul, hook] \arrow[ur, hook] & & & &
\end{tikzcd}
$$

Similarly,

$$\begin{array}{c} \mathrm{GInj}(R) \\ \uparrow \\ ?\ \mathcal{C} \\ \uparrow \\ \mathrm{WGInj}(R) \\ \uparrow \\ \mathrm{GInj}(R) \end{array}$$

Then, $\mathrm{PGF}(R) = \mathrm{GProj}(R) = \mathrm{WGProj}(R) = \mathcal{A}$,

$\mathrm{GFlat}(R) = \mathrm{WGFlat}(R) = \mathcal{B}$ and

$\mathrm{GInj}(R) = \mathrm{WGInj}(R) = \mathcal{C}$

Some more definitions:

Let R be a commutative ring and G be a group.

G is said to be of type Φ_R if it has the property that for every RG-module M:
" $\mathrm{pd}_{RG} M < +\infty \iff \mathrm{pd}_{RH} M < +\infty \quad \forall\, H$ finite $\leq G$ "

Let $B(G,R)$ be the RG-module which consists of all functions $G \to R$ whose image is a finite subset of R. Then, an RG-module M is called **Benson's cofibrant** if $M \otimes_R B(G,R)$ is a projective RG-module.

A **characteristic module** for G over R is an R-projective RG-module A with $\mathrm{pd}_{RG} A < +\infty$, which admits an R-split RG-linear monomorphism $i : R \to A$

A **weak characteristic module** for G over R is an R-flat RG-module A with $\mathrm{fd}_{RG} A < +\infty$, which admits an R-pure RG-linear monomorphism $j : R \to A$

Kropholler's hierarchy:

Let $\mathcal{X}$ be any class of groups. We define $H_0\mathcal{X} := \mathcal{X}$ and $\forall$ ordinal number $\alpha > 0$ a group G belongs to $H_\alpha\mathcal{X}$ iff there exists a finite dimensionae contractible CW-complex on which G acts such that every isotropy subgroup of the action belongs to $H_\beta\mathcal{X}$ for some ordinal $\beta < \alpha$. We say that a group G belongs to $H\mathcal{X}$ iff $\exists$ an ordinal α such that G belongs to $H_\alpha\mathcal{X}$. Moreover, we define a group G to be in $\mathcal{L}H\mathcal{X}$ iff all finitely generated subroups of G are in $H\mathcal{X}$.

$$\boxed{\begin{array}{c} \mathcal{F} = \text{Finite groups} \subseteq \Phi_R \subseteq \mathfrak{X}_{Char} \subseteq \mathfrak{X}_{WChar} \\ \mathcal{LHF} \subseteq \mathcal{LH}\Phi_R \subseteq \mathcal{LH}\mathfrak{X}_{Char} \subseteq \mathcal{LH}\mathfrak{X}_{wchar} \end{array}}$$

<u>A result of R. Biswas (2021)</u>:

Let R be a commutative ring of $\mathrm{gl.dim} R < +\infty$ and G be an $\mathcal{LHF}$ group or a group of type Φ_R. Then:

$$\boxed{\mathrm{WGProj}(RG) = \mathrm{GProj}(RG) = \mathrm{CoF}(RG)}$$

(S. 2024) $\boxed{\boxed{\mathrm{CoF}(RG)} \leq \mathrm{PGF}(RG)}$ $\forall$ commutative ring R, $\forall$ group G

$\mathrm{CoF}(RG) = \mathcal{A}$

$\Downarrow$

Theorem (S.2024): Let R be a commutative ring of $\mathrm{gl.dim} R < +\infty$ and G be an LHF group or a group of type Φ_R. Then:

$$\boxed{\mathrm{WGProj}(RG) = \mathrm{GProj}(RG) = \mathrm{CoF}(RG) = \mathrm{PGF}(RG)}$$

We generalise this result in two directions:

- we relax the assumptions concerning the commutative ring R
- we extend our study to broader classes of groups

For this purpose, we introduced Gorenstein analogues of the class of Benson's cofibrants.

Theorem 1 (S. 2024):

Let R be a commutative ring such that $\mathrm{sfli}\, R < +\infty$ and G be an $\mathcal{LH}\mathfrak{F}_{\mathrm{wchar}}$ - group.
Then:

1) $\mathcal{X}_{B,\mathrm{PGF}} = \mathrm{PGF}(RG) = \mathrm{WGProj}(RG) = \mathrm{GProj}(RG)$

$= \mathcal{A}$

where $\mathcal{X}_{B,\mathrm{PGF}} = \{ M \in \mathrm{Mod}(RG) : M \otimes_R B(G,R) \in \mathrm{PGF}(RG) \}$

and hence every Gorenstein projective RG - module is Gorenstein flat!

2) $\mathcal{X}_{B,GFlat} = WGFlat(RG) = GFlat(RG)$ $\mathcal{B}$

where $\mathcal{X}_{B,GFlat} = \{M \in Mod(RG) : M \otimes_R B(G,R) \in GFlat(RG)\}$

Šaroch and Šťovíček (2020) : $\mathfrak{PGF} = (PGF, PGF^{\perp})$ is a complete hereditary cotorsion pair.

Corollary 1 (S. 2024) :

Let R be a commutative ring such that $sfli R < +\infty$ and G be an $\mathcal{LH}\mathfrak{F}_{wchar}$-group. Then:

$(GProj(RG), GProj(RG)^{\perp})$ is a complete hereditary cotorsion pair

Theorem 2 (S. 2024):

Let R be a commutative ring such that $\mathrm{spli}\, R < +\infty$ and G be an $\mathcal{LH}\mathfrak{F}_{char}$ - group.

Then:

$$\mathcal{Y}_{B,\mathrm{GInj}} = \mathrm{WGInj}(RG) = \mathrm{GInj}(RG)$$

where $\mathcal{Y}_{B,\mathrm{GInj}} = \{ M \in \mathrm{Mod}(RG) : \mathrm{Hom}_R(B(G,R), M) \in \mathrm{GInj}(RG) \}$

Corollary 2 (S. 2024):

(i) Let R be a commutative ring such that $\operatorname{sfli} R < +\infty$. Then, Gorenstein flatness and Gorenstein projectivity are closed under subgroups that are in $\mathcal{LH}\mathfrak{F}_{\mathrm{wchar}}$.

(ii) Let R be a commutative ring such that $\operatorname{spli} R < +\infty$. Then, Gorenstein injectivity is closed under subgroups that are in $\mathcal{LH}\mathfrak{F}_{\mathrm{char}}$.

One of the open problems in the area asks whether the tensor product of Gorenstein projective modules (resp., Gorenstein flat modules) is Gorenstein projective (resp., Gorenstein flat).

Similarly, it is an open problem whether the group of homomorphisms between a Gorenstein projective and a Gorenstein injective module is Gorenstein injective.

The following results are crucial in our characterizations in Theorems 1 and 2 :

Theorem 1' (S. 2024):

Let R be a commutative ring such that $\mathrm{sfli}\,R < +\infty$ and G be an $\mathcal{LH}\mathfrak{X}_{\mathrm{wchar}}$-group. Then:

1) For every (weak) Gorenstein projective RG-module M and every RG-module N which is projective as R-module, the RG-module $M \otimes_R N$ is Gorenstein projective.

$$\mathrm{WGProj}(RG) \otimes_R \mathrm{Proj}(R) \subseteq \mathrm{GProj}(RG)$$

2) For every (weak) Gorenstein flat RG-module M and every RG-module N which is flat as R-module, the RG-module $M \otimes_R N$ is Gorenstein flat.

$$\mathrm{WGFlat}(RG) \otimes_R \mathrm{Flat}(R) \subseteq \mathrm{GFlat}(RG)$$

Theorem 2' (S. 2024):

Let R be a commutative ring such that $\operatorname{spli} R < +\infty$ and G be an $\mathcal{LH}\mathfrak{F}_{char}$-group.
Then, for every (weak) Gorenstein injective RG-module M and every RG-module N which is projective as R-module, the RG-module $\operatorname{Hom}_R(N, M)$ is Gorenstein injective.

$$\operatorname{Hom}_R(\operatorname{Proj}(R), \operatorname{WGInj}(RG)) \subseteq \operatorname{GInj}(RG)$$

Useful tools for our proofs:

- Transfinite induction
- Ideas from the work of Cornick and Kropholler
- Our elegant estimates :
  - $\max\{\mathrm{Gcd}_R G, \mathrm{spli} R\} \leq \mathrm{spli}(RG) \leq \mathrm{Gcd}_R G + \mathrm{spli} R$

which yields the nice estimates

$$\max\{\mathrm{Gcd}_R G, \mathrm{Ggl.dim} R\} \leq \mathrm{Ggl.dim}(RG) \leq \mathrm{Gcd}_R G + \mathrm{Ggl.dim} R$$

(S. 2024)

generalizing in this Gorenstein setting the classical estimates of $gl.dim(RG)$:

$$\max\{cd_R G, gl.dim R\} \leq gl.dim(RG) \leq cd_R G + gl.dim R$$

- $\max\{Ghd_R G, sfli R\} \leq sfli(RG) \leq Ghd_R G + sfli R$

which yields the nice estimates

$$\max\{Ghd_R G, Gwgl.dim R\} \leq Gwgl.dim(RG) \leq Ghd_R G + Gwgl.dim R$$

(Kaperonis, S. 2025)

References:

[1] Biswas, R.: Benson's cofibrants, Gorenstein projectives and a related conjecture, Proc. Edinb. Math. Soc (2) 64, no. 4, 779-799, (2021)

[2] Cornick, J., Kropholler, P.H.: On complete resolutions Topol. Appl. 78, 235-250 (1997)

[3] Cornick, J., Kropholler, P.H.: Homological finiteness conditions for modules over group algebras. J. London Math. Soc. 58, 49-62 (1998)

[4] Kaperonis I., Stergiopoulou D.-D.: Finiteness criteria for Gorenstein homological dimension and some invariants of Groups. Commun. Algebra, 1-19, (2025), https://doi.org/10.1080/00927872.2025.2465908.

[5] Šaroch, J., Šťovíček, J.: Singular compactness and definability for Σ-cotorsion and Gorenstein modules. Sel.

Math. New Ser. 26, 23 (2020)

[6] Stergiopoulou, D.-D.: Projectively coresolved Gorenstein flat dimension of groups. J. Algebra 641:105-146, (2024)

[7] Stergiopoulou, D.-D.: Gorenstein modules and dimension over large families of infinite groups. Collect. Math. 1-22, https://doi.org/10.1007/s13348-024-00454-8.

Thank you!