Gorenstein modules and dimension over large families of infinite groups

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The talk is based on the paper:

Stergiopoulou, D.-D.: Govenstein modules and dimension over large families of infinite groups. Collectanea Mathematica (to appear)

Contents:

- · Overriew and a central question
- · Basic definitions
- Theorems
- Some usefull tools
- References

Gorenstein homological algebra is the relative homological algebra based on Gorenstein projective, Gorenstein injective and Gorenstein flat modules.

In classical homological algebra every projective module is flat.

<u>Question</u>: Is the class of Govenstein projective modules contained in the class of Govenstein flat modules?

In this talk, we consider the case where the base ring is a group ring.

Govenstein modules: Let R be a ving.

A module M is Govenstein projective if it is a syzygy of an acyclic complex $P = \cdots \rightarrow P_{i+1} \rightarrow P_i \rightarrow P_{i-1} \rightarrow \cdots$

of projectives such that the complex $Hom_R(P,Q)$ is acyclic for every projective module Q.

A module M is Govenstein flat if it is a syzygy of an acyclic complex $F = \cdots \to F_{i+1} \to F_{i-1} \to \cdots$ of flats such that the complex $I \otimes F$ is acyclic for every injective module I.

A module M is Govenstein injective if it is a syzygy of an acyclic complex

$$T = \cdots \rightarrow I_{i+1} \rightarrow I_i \rightarrow I_{i-1} \rightarrow \cdots$$

of injectives such that the complex $Hom_R(J, I)$ is acyclic for every injective module J.

A module M is projectively coresolved Govenstein flat if it is a syzygy of an acyclic complex

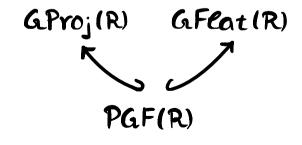
$$P = \cdots \rightarrow P_{i+1} \rightarrow P_i \rightarrow P_{i-1} \rightarrow \cdots$$

of projectives such that the complex $I \otimes P$ is acyclic for every injective module I.

Projectively covesolved Govenstein flat modules (PGF) were introduced by Saroch and Stovicek (2020)

They showed that every PGF module is Govenstein projective.

It is clear that every PGF module is Govenstein flat.



<u>Question</u>: Is the class of Govenstein projective modules contained in the class of Govenstein flat modules?



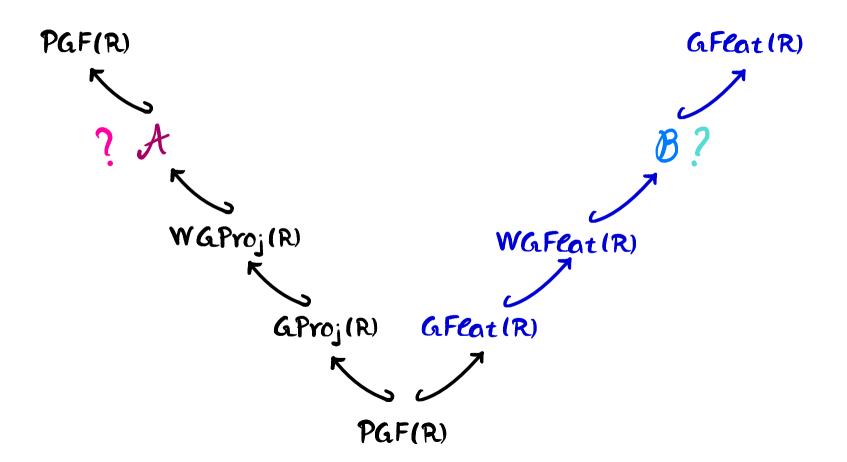
<u>Question</u>: Is the class of Govenstein projective modules equal to the class of projectively covesolved Govenstein flat modules?

The questions are equivalent!

A module M is weak Govenstein projective if it is a syzygy of an acyclic complex $P = \cdots \rightarrow P_{i+1} \rightarrow P_i \rightarrow P_{i-1} \rightarrow \cdots$ of projectives.

A module M is weak Govenstein injective if it is a syzygy of an acyclic complex $T = \cdots \to T_{i+1} \to T_i \to T_{i-1} \to \cdots$ of injectives.

A module M is weak Govenstein flat if it is a syzygy of an acyclic complex $F = \cdots \to F_{i+1} \to F_i \to F_{i-1} \to \cdots$ of flats.



Similarly, GInj(R) WGInj(R) GInj(R)

Then, PGF(R) = GProj(R) = WGProj(R) = A, GF(at(R) = WGF(at(R) = B) and GInj(R) = WGInj(R) = C

Some more definitions:

Let R be a commutative ring and G be a group.

G is said to be of type Φ_R if it has the property that for every RG-module M:

"pdRGM <+00

pdRHM <+00

H finite \(\le \text{G} \) "

Let B(G,R) be the RG-module which consists of all functions $G \rightarrow R$ whose image is a finite subset of R. Then, an RG-module M is called Benson's cofibrant if $M \otimes B(G,R)$ is a projective RG-module.

A characteristic module for a over R is an R-projective RG-module A with $pol_{RG}A < t\infty$, which admits an R-split RG-linear monomorphism $i: R \rightarrow A$

A weak characteristic module for a over R is an R-flat RG-module A with $fol_{RG}A < t\infty$, which admits an R-pure RG-linear monomorphism $j: R \rightarrow A$

Kropholler's hierarchy:

Let X be any class of groups. We define $H_0X := X$ and Vordinal number a>0 a group & belongs to Hax iff there exists a finite dimensional contractible CW-complex on which G acts such that every isotropy subgroup of the action belongs to Hox for some ordinal B<a. We say that a group G belongs to HX iff I an ordinal a such that G belongs to Hax. Moreover, we define a group a to be in SHX iff all finitely generated subroups of G are in HX.

$$F = Finite groups \subseteq \Phi_R \subseteq \mathcal{X}_{char} \subseteq \mathcal{X}_{wchar}$$

$$LHF \subseteq LH\Phi_R \subseteq LH\mathcal{X}_{char} \subseteq LH\mathcal{X}_{wchar}$$

A result of R. Biswas (2021):

Let R be a commutative ring of gl.dimR<+00 and G be an LHF group or a group of type Φ_{R} . Then:

(S. 2024) Cof(RG) = PGF(RG)

II

A

V commutative ring R V group G

Theorem (S.2024): Let R be a commutative ring of gl.dimR (+00 and G be an LHF group or a group of type Φ_R . Then:

WGProj (RG) = GProj (RG) = CoF(RG) = PGF(RG)

We generalise this result in two directions:

- we relax the assumptions concerning the commutative ring R
- we extend our study to broader classes of groups

For this purpose, we introduced Gorenstein analogues of the class of Benson's cofibrants.

Theorem 1 (5.2024):

Let R be a commutative ring such that sfliR < +00 and G be an LHXwchar - group.
Then:

Then:

1) $\chi_{B,PGF} = PGF(RG) = WGProj(RG) = GProj(RG)$

where $\chi_{B,PGF} = \{ M \in Mod(RG) : M \otimes B(G,R) \in PGF(RG) \}$

and hence every Gorenstein projective RG-module is Gorenstein flat!

2) $\chi_{B,GFlat} = WGFlat(RG) = GFlat(RG)$ where $\chi_{B,GFlat} = \{ M \in Mod(RG) : M \otimes B(G,R) \in GFlat(RG) \}$

Saroch and Štovíček (2020): FBF = (PGF, PGF¹) is a complete hereditary cotorsion pair.

Corollary 1 (S. 2024):

Let R be a commutative ring such that sfliR <+00 and G be an LHXwchar - group. Then:

(GProj(RG), GProj(RG)¹) is a complete hereditary cotorsion pair

Theorem 2 (5.2024):

Let R be a commutative ring such that splik < +00 and G be an LHX_{char} - group.

Then:

$$y_{B,GInj} = WGInj(RG) = GInj(RG)$$

where $y_{B,GInj} = \{M \in Mod(RG) : Hom_{R}(B(G,R),M) \in GInj(RG)\}$

Corollary 2 (5. 2024):

- (i) Let R be a commutative ring such that sfliR < +00. Then, Govenstein flatness and Govenstein projectivity are closed under subgroups that are in LHXwchar.
- (ii) Let R be a commutative ring such that splik < +00. Then, Governstein injectivity is closed under subgroups that are in LHXchar.

One of the open problems in the area asks whether the tensor product of Gorenstein projective modules (resp., Gorenstein flat modules) is Gorenstein projective (resp., Gorenstein flat).

Similarly, it is an open problem whether the group of homomorphisms between a Gorenstein projective and a Gorenstein injective module is Gorenstein injective.

The following results are crucial in our characterizations in Theorems 1 and 2:

Theorem 1'(5.2024):

Let R be a commutative ring such that sfliR < +00 and G be an LHXwchar - group. Then:

1) For every (weak) Govenstein projective RG-module M and every RG-module N which is projective as R-module, the RG-module M&N is Govenstein projective.

WGProj (RG) ⊗ Proj (R) = GProj (RG)

2) For every (weak) Govenstein flat RG-module M and every RG-module N which is flat as R-module, the RG-module M&N is Govenstein flat.

Waflat (RG) & Flat(R) = aflat (RG)

Theorem 2'(5.2024):

Let R be a commutative ring such that spliR < +00 and G be an LHX_{char} - group.

Then, for every (weak) Gorenstein injective RG-module M and every RG-module N which is projective as R-module, the RG-module Hom_R(N,M) is Gorenstein injective.

Homa (Proj (R), WGInj (RG)) = GInj (RG)

Useful tools for our proofs:

- ► Transfinite induction
- Ideas from the work of Cornick and Kropholler
- Dur elegant estimates:
 - max {Gcd_RG, spliR} ≤ spli(RG) ≤ Gcd_RG + spliR
 which yields the nice estimates

 $\max\{Gcd_RG, Ggl.dimR\} \leq Ggl.dim(RG) \leq Gcd_RG + Ggl.dimR$ (S. 2024)

generalizing in this Gorenstein setting the classical estimates of gl.dim(RG):

 $\max\{cd_RG, gl.dimR\} \leq gl.dim(RG) \leq cd_RG + gl.dimR$

max {Ghd_RG, sfliR} ≤ sfli(RG) ≤ Ghd_RG + sfliR
 which yields the nice estimates

 $\max\{Ghd_RG, Gwgl.dimR\} \leq Gwgl.dim(RG) \leq Ghd_RG + Gwgl.dimR$ (Kaperonis, S. 2025)

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Thank you!