On the action of the symmetric group on the free LAnKe

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10th Greek Algebra and Number Theory Conference
University of Crete, Heraclion, 23-24 May, 2025

Acknowledgements: The research project is implemented in the framework of H.F.R.I. Call "Basic research Financing Horizontal support of all Sciences" under the National Recovery and Resilience Plan "Greece 2.0" funded by the European Union Next Generation EU, H.F.R.I. Project Number: 14907.

Greece 2.0 NATIONAL RECOVERY AND RESILIENCE PLAN

Funded by the European Union NextGenerationEU

H.F.R.I. Hellenic Foundation for Research & Innovation

(Joint work with M. Maliakas)

Let $\mathbb{K}$ be a field of characteristic zero.

<u>Definition</u>: A Lie algebra of the n-th kind (LAnKe or Filippov algebra) is a $\mathbb{K}$-vector space $\mathcal{L}$ equipped with an n-linear bracket $[-,-,\dots,-] : \mathcal{L}^n \to \mathcal{L}$ such that $\forall\, x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_{n-1} \in \mathcal{L}$

1) $[x_1, x_2, \dots, x_n] = \mathrm{sign}(\sigma) \cdot [x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}] \quad \forall \sigma \in \mathfrak{S}_n$

2) Generalized Jacobi Identity

$$[[x_1, x_2, \dots, x_n], y_1, y_2, \dots, y_{n-1}]$$
$$= \sum_{i=1}^{n} [x_1, x_2, \dots, x_{i-1}, [x_i, y_1, y_2, \dots, y_{n-1}], x_{i+1}, \dots, x_n]$$

examples:

$$[[x_1, x_2], y_1] = [[x_1, y_1], x_2] + [x_1, [x_2, y_1]] \quad (n=2)$$

$$[[x_1, x_2, x_3], y_1, y_2] = [[x_1, y_1, y_2], x_2, x_3] +$$
$$+ [x_1, [x_2, y_1, y_2], x_3] + [x_1, x_2, [x_3, y_1, y_2]] \quad (n=3)$$

Clearly a Lie algebra of the 2-kind is an ordinary Lie algebra.

Filippov algebras were found independently by mathematicians and physicists (more than once since Filippov 1986)

Definition: The free LAnKe on a set X is a LAnKe $\mathcal{L}$ together with a map $i : X \to \mathcal{L}$ such that if $f : X \to \mathcal{L}'$ is a map, where $\mathcal{L}'$ is a LAnKe, then there is a unique LAnKe homomorphism $F : \mathcal{L} \to \mathcal{L}'$ such that $f = F \circ i$.

Free LAnKes on X are isomorphic.

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & \mathcal{L} \arrow[d, "F"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathcal{L}' \end{tikzcd}$$

For $n = 2$, the free LAnKe is the free Lie algebra.

We study the action of the symmetric group $\mathfrak{S}_m$ on the multilinear component of free LAnKe.

Definition: The multilinear component $\mathrm{Lie}_n(m)$ of the free LAnKe on $[m] := \{1, 2, \ldots, m\}$ is spanned by the bracketed words on $[m]$ in which each i appears exactly once.

Each bracketed word has the same number of brackets, say k, and $m = (n-1) \cdot k + 1$.

For example, the bracketed permutations

$[[1,2,3],4,5]$, $[[1,2,4],3,5]$, $[[1,2,5],3,4]$, $[[1,3,4],2,5]$,

$[[1,3,5],2,4]$, $[[1,4,5],2,3]$, $[[2,3,4],1,5]$, $[[2,3,5],1,4]$,

$[[2,4,5],1,3]$, $[[3,4,5],1,2]$

span the vector space $\mathrm{Lie}_3(5)$.

GJI : $[[1,2,3],4,5] = [[1,4,5],2,3] + [1,[2,4,5],3] + [1,2,[3,4,5]]$

$= [[1,4,5],2,3] - [[2,4,5],1,3] + [[3,4,5],1,2]$

examples:

$n=2,\ k=4,\ m=5$ $\quad [[[1,2],[3,4]],5]$

$n=3,\ k=4,\ m=9$ $\quad [[1,2,3],[[4,5,6],7,8],9]$

$n=5,\ k=3,\ m=13$ $\quad [[[1,2,3,4,5],6,7,8,9],10,11,12,13]$

nested bracketed word

$n=4,\ k=5,\ m=16$ $\quad [[1,2,3,4],[5,6,7,8],[9,10,11,12],[13,14,15,16]]$

We consider the action of $\mathfrak{S}_m$ on $\mathrm{Lie}_n(m)$ given by replacing i by $\sigma(i)$ in each bracketed word and denote the corresponding representation by $\rho_{n,k}$.

Theorem: Every bracketed word is a linear combination of nested bracketed words [FHW]

Let λ be a partition of m and S^λ the corresponding Specht module of the symmetric group $\mathfrak{S}_m$.

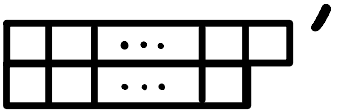
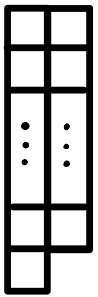
$\{S^\lambda \mid \lambda \vdash m\}$ = complete set of inequivalent irreducible representations of $\mathfrak{S}_m$

ex.: $S^{(m)}$ = trivial representation

$S^{(1,1,\dots,1)}$ = sign representation

Friedmann, Hanlon, Stanley and Wachs (2021) showed that

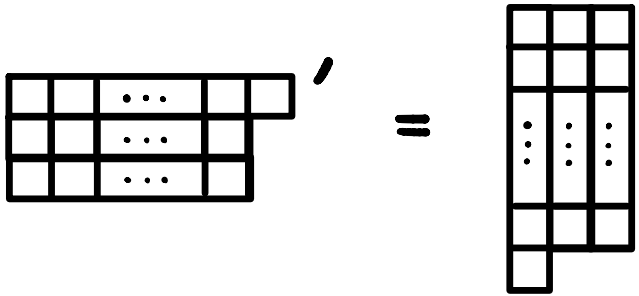
Theorem 1: $\rho_{n,2} \cong S^{(n,n-1)'}, \quad n \geq 2$ [FHSW]

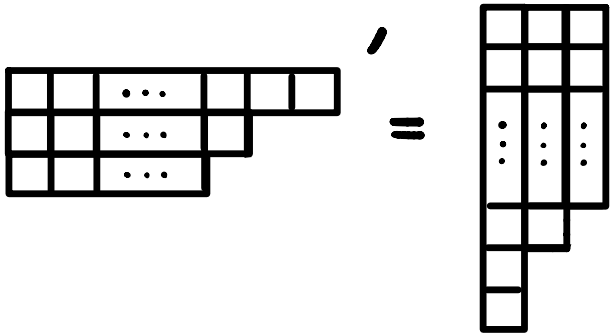
$(n,n-1)' =$  $' =$  $= (2^{n-1}, 1)$

They stated without proof that:

Theorem 2: $\rho_{n,3} \cong S^{(n,n-1,n-1)'} \oplus S^{(n+1,n-1,n-2)'}$

[MS], [FHW]

$(n, n-1, n-1)' =$  $= (3^{n-1}, 1)$

$(n+1, n-1, n-2)' =$  $= (3^{n-1}, 2, 1^2)$

Maliakas and S. (2024-2025)[MS] gave the first published proof of Theorem 2, using polynomial representations of $GL_N(\mathbb{K})$ and algebraic combinatorics of semistandard tableaux.

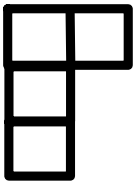
What was known about $\rho_{n,k}$?

- The case $n=2$, by Kraskiewicz and Weyman (2001)

Let i and m relative prime and λ a partition of m. The multiplicity of the irreducible representation indexed by λ in the free Lie algebra on m generators is equal to the number of standard Young tableaux of shape λ and of major index congruent to $i \pmod{m}$.

<u>Definition</u>: In a standard Young tableau, the integer k is a descent if $k+1$ appears in a row strictly below k. The sum of the descents is called the major index of the tableau.

examples:

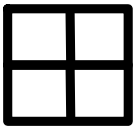
1) $i = 3$, $m = 4$ and $\lambda = (2, 1^2) =$ 

standard Young tableaux

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

major index $\quad 2+3 \quad 1+3 \quad 1+2 \equiv 3 \pmod 4$

$\times \quad \times \quad \checkmark$

2) $i = 3$, $m = 4$ and $\lambda = (2, 2) =$ 

standard Young tableaux	$\begin{array}{\|c\|c\|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}$
major index	2	1+3
	×	×

- The case $k=2$, by Friedmann, Hanlon, Stanley and Wachs (2021)

$$\rho_{n,2} \cong S^{(2^{n-1},1)}, \quad \dim \rho_{n,2} = \frac{1}{n+1}\binom{2n}{n}$$

the n-th Catalan number

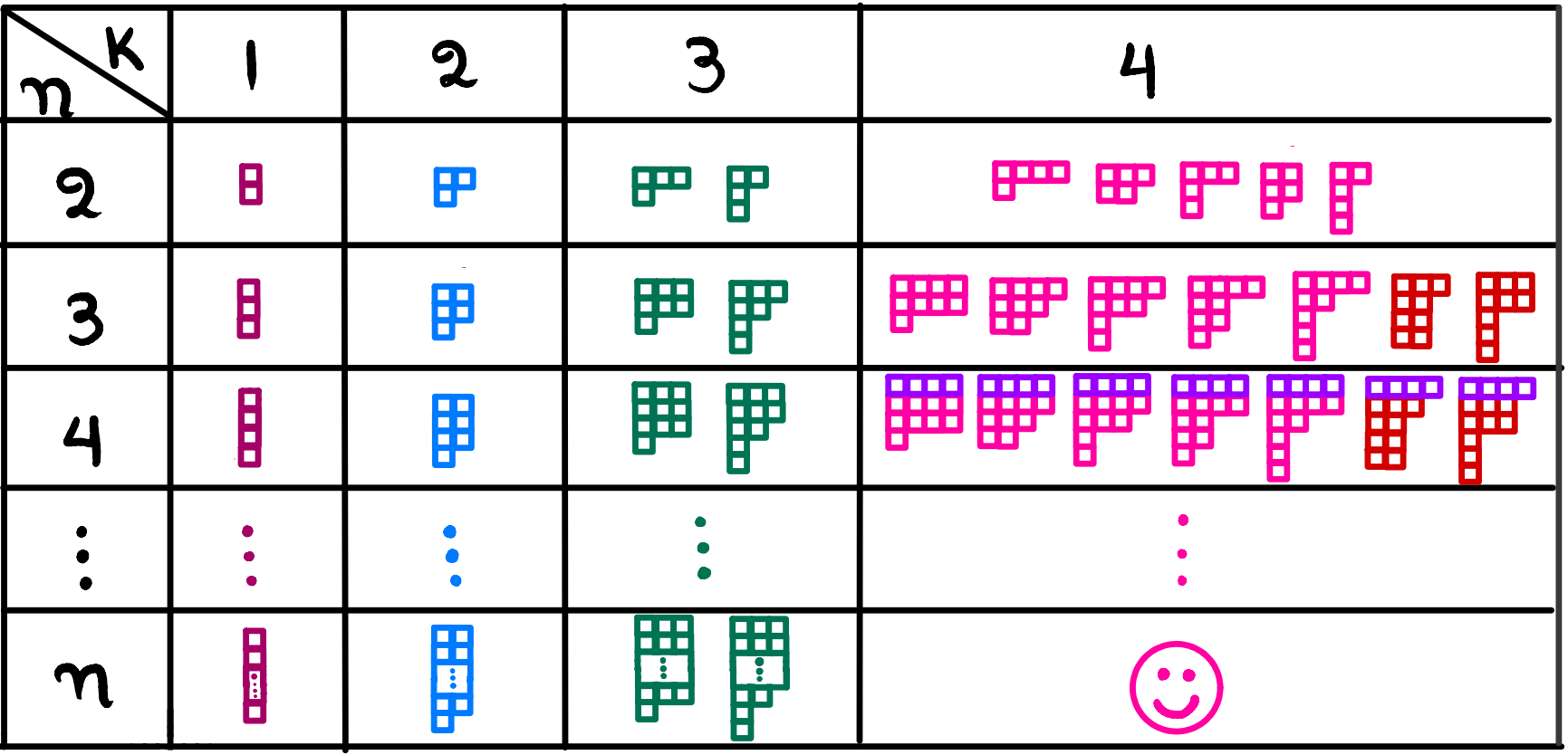
$n \backslash k$	1	2	3	4
2	1^2	21	$31 \oplus 21^2$	$41 \oplus 32 \oplus 31^2 \oplus 2^21 \oplus 21^3$
3	1^3	2^21	$3^21 \oplus 321^2$	$4^21 \oplus 432 \oplus 431^2 \oplus 42^21 \oplus 421^3 \oplus 32^3 \oplus 3^21^3$
4	1^4	2^31	$3^31 \oplus 3^221^2$	?
$\vdots$	$\vdots$	$\vdots$	$\vdots$	?
n	1^n	$2^{n-1}1$	$3^{n-1}1 \oplus 3^{n-2}21^2$	?

sign representation

Theorem 1 [FHSW]

Theorem 2 [MS], [FHW]

[MS], [FHW]

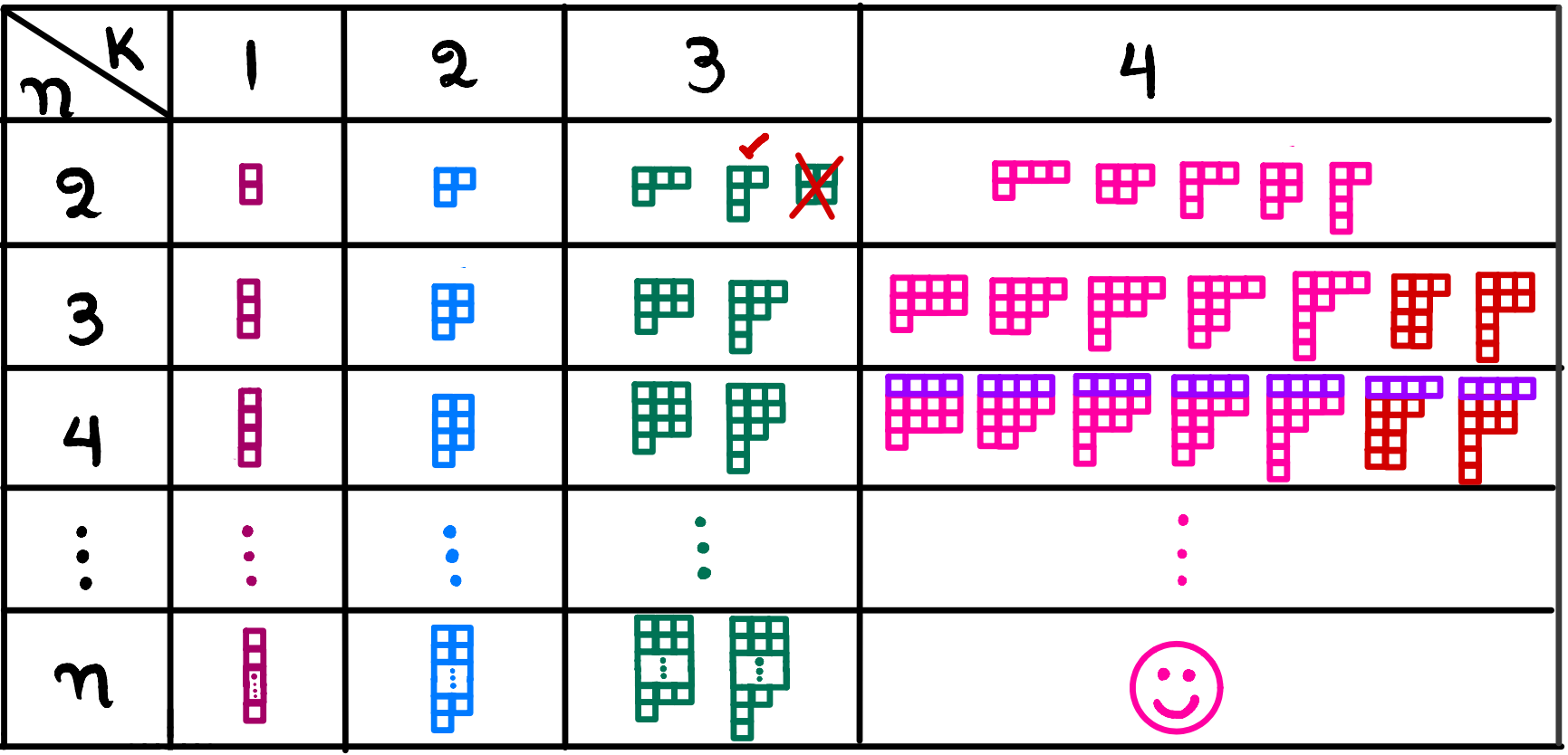


sign representation

Theorem 1 [FHSW]

Theorem 2 [MS], [FHW]

[MS], [FHW]



sign representation

Theorem 1 [FHSW]

Theorem 2 [MS], [FHW]

[MS], [FHW]

Main open problem: The decomposition of $\rho_{n,k}$ into a sum of irreducible representations $\forall n, k$.

Main result for Schur modules:

Theorem (Maliakas - S.): Let $N \geq 3n-2$. Then, the cokernel of the map

$$\gamma_1+\gamma_2+\gamma_3 : \Lambda(\lambda) \oplus \Lambda(\lambda) \oplus \Lambda(\nu) \longrightarrow \Lambda(\lambda)$$

is isomorphic to $L_\lambda \oplus L_\mu$ as $GL_N(\mathbb{K})$-modules.

$\lambda = (n, n-1, n-1)$, $\mu = (n+1, n-1, n-2)$, $\nu = (n, n, n-2)$

L_λ = the irreducible polynomial representation of corresponding to a partition λ of m.

$\{L_\lambda \mid \lambda \vdash m\}$ = complete set of inequivalent irreducible polynomial representations of weight m

example: $L_{(\alpha)} = \Lambda^\alpha$, $L_{(1^\alpha)} = S_\alpha$

Our proof is based on combinatorics of semistandard tableaux.

Setup for our proof for $k=3$:

We find a presentation of $\rho_{n,3}$ with generators and relations using the Generalised Jacobi Identity.

$\rho_{n,3}$ is generated by the nested brackets

$$[[[x_1, x_2, \ldots, x_n], y_1, y_2, \ldots, y_{n-1}], z_1, z_2, \ldots, z_{n-1}]$$

Let $M = \Lambda(\lambda)$ where $\lambda = (n, n-1, n-1)$.

We define the following GL-maps

$$\begin{tikzcd} M \arrow[dr, "\gamma_1"] & \\ & M \\ M \arrow[ur, "\gamma_2"'] & \end{tikzcd}$$

$$\gamma_1 \longleftrightarrow \begin{array}{|c|c|c|c|}\hline 1 & \cdots & 1 & 1 \\ \hline 2 & \cdots & 2 \\ \cline{1-3} 3 & \cdots & 3 \\ \cline{1-3}\end{array} + (-1)^n \cdot \begin{array}{|c|c|c|c|}\hline 1 & 2 \cdots & 2 & 2 \\ \hline 1 & \cdots & 1 \\ \cline{1-3} 3 & \cdots & 3 \\ \cline{1-3}\end{array}$$

$$\gamma_2 \longleftrightarrow \begin{array}{|c|c|c|c|}\hline 1 & \cdots & 1 & 1 \\ \hline 2 & \cdots & 2 \\ \cline{1-3} 3 & \cdots & 3 \\ \cline{1-3}\end{array} + (-1)^n \cdot \begin{array}{|c|c|c|c|}\hline 1 & \cdots & 1 & 1 \\ \hline 3 & \cdots & 3 \\ \cline{1-3} 2 & \cdots & 2 \\ \cline{1-3}\end{array} + (-1)^n \cdot \begin{array}{|c|c|c|c|}\hline 2 & 3 \cdots & 3 & 3 \\ \hline 1 & 2 \cdots & 2 \\ \cline{1-3} 1 & 1 \cdots & 1 \\ \cline{1-3}\end{array}$$

(Applying the GJI twice)

Relations in $\mathrm{Lie}_n(m)$

$$[[[x_1,\dots,x_n],y_1,\dots,y_{n-1}],z_1,\dots,z_{n-1}] - \sum_{i=1}^{n}(-1)^{i-1}[[[x_i,y_1,\dots,y_{n-1}],x_1,\dots,\hat{x}_i,\dots,x_n],z_1,\dots,z_{n-1}] = 0 \qquad \text{(R1)}$$

$$[[[x_1,\dots,x_n],y_1,\dots,y_{n-1}],z_1,\dots,z_{n-1}] - [[[x_1,\dots,x_n],z_1,\dots,z_{n-1}],y_1,\dots,y_{n-1}] - \sum_{i=1}^{n-1}(-1)^{i-1}\sum_{j=1}^{n}(-1)^{j}[[[y_i,z_1,\dots,z_{n-1}],x_j,y_1,\dots,\hat{y}_i,\dots,y_{n-1}],x_1,\dots,\hat{x}_j,\dots x_n] = 0 \qquad \text{(R2)}$$

We define also the GL-map $\gamma_3 : \Lambda(V) \longrightarrow M = \Lambda(\lambda)$

$$\gamma_3 \longleftrightarrow \begin{array}{|c|c|c|c|}\hline 1 & \cdots & 1 & 1 \\ \hline 2 & 3 \cdots & 3 & \\ \cline{1-3} 2 & \cdots & 2 & \\ \cline{1-3}\end{array} + (-1)^n \cdot \begin{array}{|c|c|c|c|}\hline 2 & \cdots & 2 & 2 \\ \hline 1 & 3 \cdots & 3 & \\ \cline{1-3} 1 & \cdots & 1 & \\ \cline{1-3}\end{array}$$

The reason we consider the map γ_3 is due to the specific relation of $\mathrm{Lie}_n(m)$:

$$\boxed{\begin{aligned} &\sum_{i=1}^{n} [[[x_1, \dots, x_n], y_i, z_1, \dots, z_{n-2}], y_1, \dots, \hat{y}_i, \dots, y_n] \\ &+ \sum_{i=1}^{n} [[[y_1, \dots, y_n], x_i, z_1, \dots, z_{n-2}], x_1, \dots, \hat{x}_i, \dots, x_n] = 0 \end{aligned}}$$

$$S = \begin{matrix} 1^{(a_{11})} & 2^{(a_{12})} & 3^{(a_{13})} \\ 1^{(a_{21})} & 2^{(a_{22})} & 3^{(a_{23})} \\ 1^{(a_{31})} & 2^{(a_{32})} & 3^{(a_{33})} \end{matrix}$$

row semistandard tableau of shape $\mu = (\mu_1, \mu_2, \mu_3) \in \Lambda^+(3, r)$ and weight $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \Lambda(3, r)$

$$a_{i1} + a_{i2} + a_{i3} = \mu_i$$

$$a_{1i} + a_{2i} + a_{3i} = \alpha_i$$

$$\phi_S : \Lambda(\alpha) = \Lambda(\alpha_1, \alpha_2, \alpha_3) \xrightarrow{\Delta_1 \otimes \Delta_2 \otimes \Delta_3}$$

$$\Lambda(a_{11}, a_{21}, a_{31}) \otimes \Lambda(a_{12}, a_{22}, a_{32}) \otimes \Lambda(a_{13}, a_{23}, a_{33}) \cong$$

$$\Lambda(a_{11}, a_{12}, a_{13}) \otimes \Lambda(a_{21}, a_{22}, a_{23}) \otimes \Lambda(a_{31}, a_{32}, a_{33})$$

$$\xrightarrow{m_1 \otimes m_2 \otimes m_3} \Lambda(\mu_1, \mu_2, \mu_3) = \Lambda(\mu)$$

The Schur functor: is a functor f from the category of homogeneous polynomial representations of $GL_N(\mathbb{K})$ of degree m to the category of left $\mathcal{G}_m$ modules. For M an object in the first category, $f(M)$ is the weight subspace M_α where $\alpha = (1^m, 0^{N-m})$ and for $\theta : M \to N$ a morphism in the first category $f(\theta)$ is the restriction $M_\alpha \longrightarrow N_\alpha$.

examples: $f(L_\nu) = S^{\nu'}$ and $f(\Lambda^\nu) = \widetilde{M}^{\nu'}$

the space of column tabloids

The functor f is exact.

Since $\operatorname{char} \mathbb{K} = 0$, f is a category equivalence.

We show that $\boxed{f(\operatorname{coker}(\gamma_1 + \gamma_2 + \gamma_3)) \cong \rho_{n,3}}$

Main Theorem [MS] $\cong$

$$f(L_\lambda) \oplus f(L_\mu)$$

$$\|$$

$$S^{\lambda'} \oplus S^{\mu'}$$

Ideally, we want a spectral analysis of the operators γ_1, γ_2 and γ_3. But it suffices to find the irreducible summands of $M = \Lambda(\lambda)$ with eigenvalue $\neq 0$. We do this with combinatorics of semistandar tableaux. There are exactly two summands with the above property, so that

$$\boxed{\operatorname{coker}(\gamma_1 + \gamma_2 + \gamma_3) \cong L_\lambda \oplus L_\mu}$$

We show that :

1) both L_λ and L_μ appear with multiplicity one in $\operatorname{coker}(\sigma_1+\sigma_2+\sigma_3)$

2) $\forall\, \nu \neq \lambda, \mu$ the multiplicity of L_ν in $\operatorname{coker}(\sigma_1+\sigma_2+\sigma_3)$ is zero

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<u>Definition</u>: Let W be the $\mathbb{G}_m$-module generated by the nested bracketed words :

$$[[[x_1,\dots,x_n],y_1,\dots,y_{n-1}],z_1,\dots,z_n]$$

Corollary: For $m = 3\cdot(n-1)+1$ and $n \geq 2$ we have

$$\mathrm{Lie}_n(m) \cong W \big/ \mathrm{span}((R1),(R2))$$

as G_m-modules.

References:

[FHSW] Friedmann T., Hanlon P., Stanley R.P. and Wachs M.L., On a generalization of Lie(k): a CataLAnke theorem, Adv. Math. 380 (2021), 107570

[MS] Maliakas M. and Stergiopoulou D.-D., On the action of the symmetric group on the free LAnke, arXiv:2401.09405.

[FHW] Friedmann T., Hanlon P. and Wachs M.L., On an n-ary generalization of the Lie representation and tree Specht modules, arXiv:2402.19174.

Thank you

Καλό καλοκαίρι