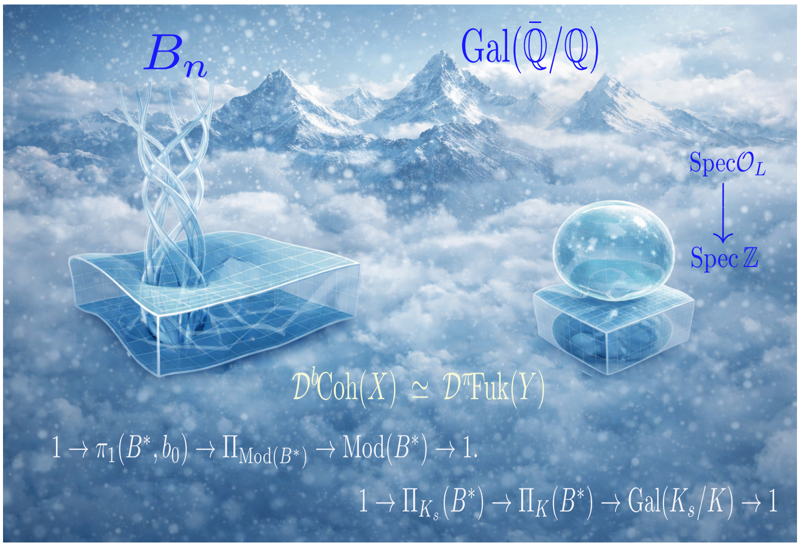
Group Actions on Algebraic Curves
GAAC

Arithmetic Topology and Topological Actions



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Contents

1. Arithmetic Topology

Arithmetic topology is concerned with the similarities between several notions and theorems in algebraic number theory and the theory of 3-manifolds. In this theory there is the Mazur-Kapranov-Reznikov dictionary (MKR-dictionary for short) and a short part of it is displayed in the following table bellow right: For a nice introduction and a detailed explanation see [36]. Arithmetic topology provides a common framework, so that representation of Braid groups and Galois representations fit together. In knot theory it is a common practice to study braid groups representations, in order to provide invariants of knots (after Markov equivalence, see [39, III.6]) and in number theory the study of Galois representations is an important tool in order to approach class field theory or more general to understand the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. Also the deformation theory of Galois representations was a crucial part of the celebrated proof of Fermat's last theorem [44], [34].

Number Theory	Topology
prime ideals	knots
ideals	links
Number Fields	3-manifolds
class group	$H_1(M,\mathbb{Z})$
Riemann's ζ-function	Selberg's ζ-function
Algebraic extensions	Ramified Top. covers
Galois groups, $\pi_1^{\mathrm{et}}(X)$	$\pi_1(X, x_0)$

Of course understanding braid groups and knots has the advantage that also 3-manifolds can be understood, either by surgery representations of 3-manifolds or by presenting them as ramified extensions of the three dimensional sphere S^3 with ramification locus a link by Alexander theorem [39, chap. IV].

The Knots–Primes Analogy. We will now describe the analogy, originating with Mazur, between knot theory in 3–manifolds and arithmetic of number fields. This analogy is based on interpreting

$$\mathrm{Spec}\mathbb{Z} \quad \text{as an arithmetic analogue of} \quad \mathbb{R}^3,$$

and closed points $\mathrm{Spec}\mathbb{F}_p \hookrightarrow \mathrm{Spec}\mathbb{Z}$ as analogues of embedded circles $S^1 \hookrightarrow \mathbb{R}^3$. Étale homotopy theory plays the role of classical homotopy theory.

In particular,

$$\pi_1^{\text{ét}}(\mathrm{Spec}\mathbb{F}_p) \cong \widehat{\mathbb{Z}}, \qquad \pi_1(S^1) \cong \mathbb{Z},$$

and $\mathrm{Spec}\mathbb{Z}$ satisfies an arithmetic form of Poincaré duality (Artin–Verdier duality), analogous to a 3–manifold.

Correspondence Between Knot Theory and Number Theory. The article establishes a systematic dictionary between topological and arithmetic objects, including:

Knot Theory	Number Theory
S^1	$\mathrm{Spec}(\mathbb{F}_q)$
$\pi_1(S^1)$	$\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$
Knot $K \subset S^3$	Prime $p \subset \mathrm{Spec}\mathbb{Z}$
Knot complement X_K	$\mathrm{Spec}(\mathcal{O}_K \setminus \{p\})$
Knot group $\pi_1(X_K)$	$\pi_1^{\text{ét}}(\mathrm{Spec}(\mathcal{O}_K \setminus \{p\}))$
Linking number	Legendre symbol
Alexander polynomial	Iwasawa polynomial

This correspondence motivates viewing number fields as branched coverings of $\mathrm{Spec}\mathbb{Z}$, analogous to branched coverings of S^3.

Knot Groups. For a knot $K \subset S^3$, let $X_K = S^3 \setminus \mathrm{int}(V_K)$ be its exterior. The *knot group* is defined as

$$G_K := \pi_1(X_K).$$

Theorem 1 (Wirtinger Presentation)**.** *Given a regular projection of an oriented knot K with arcs $c_1, \ldots, c_n$, the knot group admits a presentation*

$$G_K = \langle x_1, \ldots, x_n \mid R_1, \ldots, R_n \rangle,$$

where each relation R_i is determined by the corresponding crossing, and one relation is redundant.

As a consequence, G_K has deficiency 1. For example, the trefoil knot has group

$$G_K \cong \langle a, b \mid a^3 = b^2 \rangle \cong B_3.$$

Quadratic Reciprocity as a Linking Phenomenon. Quadratic reciprocity as the arithmetic analogue of symmetry of the linking number.

Theorem 2 (Quadratic Reciprocity)**.** *Let p, q be distinct odd primes. Then*

$$\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right)(-1)^{\frac{p-1}{2}\frac{q-1}{2}}.$$

This mirrors the symmetry

$$\mathrm{lk}(K, L) = \mathrm{lk}(L, K)$$

for oriented links in S^3. The Legendre symbol is interpreted as an arithmetic linking invariant.

Covering Spaces and Cyclic Covers. The infinite cyclic covering

$$X_\infty \to X_K$$

associated to the abelianization $G_K \twoheadrightarrow \mathbb{Z}$ plays a central role. Its deck transformation group is generated by τ.

Finite cyclic coverings

$$X_n \to X_K$$

correspond to the quotient $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$, and can be completed to branched coverings

$$M_n \to S^3$$

ramified along K via the Fox completion.

Alexander Polynomial and Iwasawa Theory. Let $H_1(X_\infty)$ be the first homology of the infinite cyclic cover. It is a module over $\mathbb{Z}[t^{\pm 1}]$, where t acts via τ.

Definition 3. The Alexander polynomial of K is

$$\Delta_K(t) = \det(t \cdot \mathrm{id} - \tau \mid H_1(X_\infty) \otimes \mathbb{Q}).$$

This construction is directly analogous to the Iwasawa module in the cyclotomic $\mathbb{Z}_p$–extension of a number field, with the Alexander polynomial corresponding to the Iwasawa polynomial.

Number Rings and Prime Decomposition. The article reformulates classical results such as Fermat's theorem on sums of two squares in terms of prime decomposition in number rings.

Proposition 4. *Let p be a prime.*

- *If $p \equiv 1 \pmod 4$, then p splits in $\mathbb{Z}[i]$:*

$$(p) = (\alpha)(\bar{\alpha}).$$

- *If $p \equiv 3 \pmod 4$, then p remains prime in $\mathbb{Z}[i]$.*

This is presented as the arithmetic analogue of decomposition of coverings and ramification in topology.

Étale Coverings. Finite étale coverings of $\mathrm{Spec}\mathcal{O}_K$ are interpreted as arithmetic analogues of unramified coverings of manifolds. The étale fundamental group

$$\pi_1^{\text{ét}}(\mathrm{Spec}\mathcal{O}_K)$$

controls these coverings in direct analogy with $\pi_1(X)$ in topology.

Fundamental / Galois groups	
Knots	**Primes**
$\pi_1(S^1) = \mathrm{Gal}(\mathbb{R}/S^1) = \langle [l] \rangle = \mathbb{Z}$ Circle $S^1 = K(\mathbb{Z}, 1)$ Loop l Universal covering $\mathbb{R}$ Cyclic covering $\mathbb{R}/n\mathbb{Z}$	$\pi_1(\mathrm{Spec}(\mathbb{F}_q)) = \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q),\ q = p^n = \langle [\sigma] \rangle = \widehat{\mathbb{Z}}$ Finite field $\mathrm{Spec}(\mathbb{F}_q) = K(\widehat{\mathbb{Z}}, 1)$ Frobenius automorphism σ Separable closure $\overline{\mathbb{F}}_q$ Cyclic extension $\mathbb{F}_{q^n}/\mathbb{F}_q$
Manifolds / Spec of a ring	
$V \simeq S^1$ $V \setminus S^1 \simeq \partial V$ ($\simeq$ denotes homotopy equivalence)	$\mathrm{Spec}(\mathcal{O}_p) \simeq \mathrm{Spec}(\mathbb{F}_q)$ $\mathrm{Spec}(\mathcal{O}_p) \setminus \mathrm{Spec}(\mathbb{F}_q) \simeq \mathrm{Spec}(k_p)$ ($\simeq$ denotes étale homotopy equivalence; $\mathcal{O}_p$ is a p-adic integer ring with residue field $\mathbb{F}_q$ and quotient field k_p)
Tubular neighborhood V	p-adic integer ring $\mathrm{Spec}(\mathcal{O}_p)$
Boundary ∂V	p-adic field $\mathrm{Spec}(k_p)$
3-manifold M	Number ring $\mathrm{Spec}(\mathcal{O}_k)$
Knot $S^1 \hookrightarrow \mathbb{R}^3 \cup \{\infty\} = S^3$	Rational prime $\mathrm{Spec}(\mathbb{F}_p) \hookrightarrow \mathrm{Spec}(\mathbb{Z}) \cup \{\infty\}$
Any connected oriented 3-manifold is a finite covering of S^3 branched along a link (Alexander's theorem)	Any number field is a finite extension of $\mathbb{Q}$ ramified over a finite set of primes
Knot group / Prime group	
$G_K = \pi_1(M \setminus K)$ $G_K \cong G_L \iff K \sim L$ for prime knots K, L	$G_{\{p\}} = \pi_1^{\mathrm{t}}(\mathrm{Spec}(\mathcal{O}_k \setminus \{p\}))$ $G_{\{p\}} \cong G_{\{q\}} \iff p = q$ for primes p, q
Linking number / Legendre symbol	
Linking number $\mathrm{lk}(L, K)$	Legendre symbol $\left(\frac{q^*}{p}\right), \quad q^* := (-1)^{\frac{q-1}{2}} q$
Symmetry $\mathrm{lk}(L, K) = \mathrm{lk}(K, L)$	Quadratic reciprocity law $\left(\frac{q}{p}\right) = \left(\frac{p}{q}\right)$ $(p, q \equiv 1 \pmod 4)$
Alexander–Fox theory / Iwasawa theory	
Infinite cyclic covering $X_\infty \to X_K$ $\mathrm{Gal}(X_\infty/X_K) = \langle \tau \rangle \simeq \mathbb{Z}$ Knot module $H_1(X_\infty)$ Alexander polynomial $\det(t \cdot \mathrm{id} - \tau \mid H_1(X_\infty) \otimes_{\mathbb{Z}} \mathbb{Q})$	Cyclotomic $\mathbb{Z}_p$-extension k_∞/k $\mathrm{Gal}(k_\infty/k) = \langle \gamma \rangle \simeq \mathbb{Z}_p$ Iwasawa module H_∞ Iwasawa polynomial $\det(T \cdot \mathrm{id} - (\gamma - 1) \mid H_\infty \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$

Table 1. Analogies between knots and primes

2. The theory of profinite braids

Y. Ihara in a series of articles [22], [23] proposed a method to treat elements in the automorphism group of the profinite free group as "profinite braids" and in this way he got a series of Galois representations similar to classical representations in the theory of Braids. We believe that this is a crucial key in understanding the deeper reasons why Arithmetic Topology works. Indeed, both the absolute Galois group and the braid groups can be seen as automorphisms of a large (profinite) free group. Also the braid groups representations are in the norm one group (see [22, prop. 2] for the definition of the norm map and the relation to cyclotomic character). Furthermore this setting is compatible with the ideas of Kapranov, Smirnov [24] on identifying the braid group as the general linear group of polynomials defined over the mythical field with one element.

The theory of profinite braids initiated by Yasutaka Ihara is a foundational program in arithmetic geometry and number theory that studies the absolute Galois group of the rational numbers ($G_{\mathbb{Q}}$) through its geometric action on the fundamental groups of algebraic varieties.This program is central to arithmetic topology, as it bridges classical knot/braid theory with deep arithmetic questions like Iwasawa theory and the Vandiver conjecture.

The Core Idea: Geometric Galois Actions. Ihara's work is a realization of Grothendieck's "Esquisse d'un Programme", which suggested that $G_{\mathbb{Q}}$ is best understood by looking at how it acts on geometric objects.

- The Arithmetic-Geometric Bridge: Just as the classical braid group acts on the fundamental group of a punctured disk, the absolute Galois group $G_{\mathbb{Q}}$ acts on the profinite fundamental group ($\hat{\pi}_1$) of the projective line minus three points, $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.
- Identifying Galois Elements: Every element $\sigma \in G_{\mathbb{Q}}$ can be identified as a pair $(\chi(\sigma), f_\sigma)$, where $\chi(\sigma)$ is the cyclotomic character (governing roots of unity) and f_σ is a "profinite word" representing the Galois action on the braid-like paths of the curve.

Key Components of the Theory.

- Profinite Braid Groups ($\hat{B}_n$): These are the profinite completions of the standard Artin braid groups. Ihara studied the continuous automorphisms of these groups to characterize $G_{\mathbb{Q}}$.
- The Grothendieck-Teichmüller Group ($\widehat{GT}$): This is a larger, mysterious group that contains $G_{\mathbb{Q}}$. Ihara and Drinfeld defined $\widehat{GT}$ by specific algebraic relations that every Galois action must satisfy. A major open problem is whether $G_{\mathbb{Q}} \cong \widehat{GT}$.
- Universal Power Series for Jacobi Sums: In a landmark 1986 paper, Ihara introduced power series that describe Galois actions on profinite fundamental groups. These series connect the geometry of braids to the values of p-adic L-functions.

2.0.1. *Profinite Braids vs. Pure Braids.* Ihara established an explicit analogy between pure braid groups (P_r) and absolute Galois groups of number fields:

Topological Issue (Pure Braid Group P_r)	**Arithmetic Analogue (Ihara Theory)**
Braid Action: $P_r \to \mathrm{Aut}(F_r)$	**Galois Action:** $G_{\mathbb{Q}} \to \mathrm{Out}(\hat{F}_r)$
Milnor Invariants: Linking numbers for pure braid links	**Arithmetic Milnor Invariants:** Defined using Galois representations
Johnson Homomorphisms: Filtrations of the automorphism group	**Ihara's Lie Algebra:** Graded $\mathbb{Z}_p$-Lie algebra of the Galois group

3. Iwasawa theory

In order to make clear this idea we will first study the following special case: In [35] C. McMullen considered unitary representations of the braid group acting on global sections of differentials of cyclic covers of the projective line. Also W. Chen in [7] motivated by the fibration of cyclic groups studied the homology $H_*(B_s, V_s)$, where $V_s = \mathbb{C}[t, t^{-1}]^{s-2}$ and the braid group B_s acts on $\mathbb{C}[t, t^{-1}]$ in terms of the Burau representation.

Let S be a compact Riemann-surface of genus g. Consider the first homology group $H_1(S, \mathbb{Z})$ which is a free $\mathbb{Z}$-module of rank $2g$. Let $H^0(S, \Omega_X)$ be the space of holomorphic differentials which is a $\mathbb{C}$-vector space of dimension g. The function

$$\begin{array}{rcl} H_1(S,\mathbb{Z}) \times H^0(X,\Omega_X) & \to & \mathbb{R} \\ \gamma, \omega & \mapsto & \langle \gamma, \omega \rangle = \mathrm{Re} \int_\gamma \omega \end{array}$$

induces a duality $H_1(S, \mathbb{Z}) \otimes \mathbb{R}$ to $H^0(X, \Omega_X)^*$, see [32, th. 5.6], [19, sec. 2.2 p. 224]. Therefore an action of a group element on $H_1(S, \mathbb{Z})$ gives rise to the contragredient action on holomorphic differentials, see also [16, p. 271].

Consider a curve Y which can be seen as Galois ramified cover $\pi : Y \to \mathbb{P}^1$ with $\Sigma = \{P_1, \ldots, P_s\}$ ramified points and with Galois group $\mathrm{Gal}(Y/\mathbb{P}^1) = G$. The open curve $Y_0 = Y - \pi^{-1}(\Sigma)$ is a topological cover of $X = \mathbb{P}^1 - \Sigma$ and can be seen as a quotient of the universal covering space $\tilde{X}$ by a free subgroup R_0 of the free group $\pi_1(X, x_0) = F_{s-1}$, where $s = \#\Sigma$.

The free group R_0 can be effectively computed using the Reidemeister-Schreier method [33, sec. 2.3 th. 2.7]. In this way we arrive at a presentation which can be also given as

$$R_0 = \langle a_1, b_1, a_2, b_2, \ldots, a_g, b_g, \gamma_1, \ldots, \gamma_s | \gamma_1\gamma_2 \cdots \gamma_s \cdot [a_1, b_1][a_2, b_2] \cdots [a_g, b_g] = 1 \rangle.$$

By the short exact sequence $1 \to R_0 \to F_{s-1} \to G \to 1$, we see that there is an action of G on R_0 modulo inner automorphisms of R_0 and in particular to a well defined action of G on $R_0/R_0' = H_1(Y_0, \mathbb{Z})$. Therefore the space $H_1(Y_0, \mathbb{Z})$ can be seen as a direct sum of indecomposable $\mathbb{Z}[G]$-modules. For example the Burau representation appears in this way.

If a subgroup of the automorphism groups of the (profinite) free group induces automorphisms on the R_0 (for instance if R is characteristic) then an action of the homology $H_1(Y_0, \mathbb{Z})$ is given.

The completed curve Y has a fundamental group which admits a presentation of the form

$$R = \langle a_1, b_1, a_2, b_2, \ldots, a_g, b_g | \cdot [a_1, b_1][a_2, b_2] \cdots [a_g, b_g] = 1 \rangle.$$

There is the following short exact sequence relating the two homology groups:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \langle \gamma_1, \ldots, \gamma_s \rangle & \longrightarrow & H_1(Y_0, \mathbb{Z}) & \longrightarrow & H_1(Y, \mathbb{Z}) & \longrightarrow & 0 \\ & & & & \downarrow \cong & & \downarrow \cong & & \\ & & & & R_0/R_0' & \longrightarrow & R/R' & & \end{array}$$

The space $\langle \gamma_1, \ldots, \gamma_s \rangle$ consists of all elements in $H_1(Y_0, \mathbb{Z})$ fixed by some element $1 \neq g \in G$. Indeed, the action of the automorphism group of a compact Riemann surface on the homology is faithful [16, V.3 p.261] therefore if $a \in H_0(Y_0, \mathbb{Z})$ is fixed by $g \neq 1$ then $a \in \langle \gamma_1, \ldots, \gamma_s \rangle$.

This method can be applied to several curves defined over a field of characteristic zero and also for curves defined over fields of positive characteristic, whenever we have a theory of fundamental groups and covering space theory available (we can use either étale fundamental groups or Mumford curves [38], [8], [9] and Berkovitch spaces [4]).

Notice that there is a variety of groups that can act on a Riemann surface/algebraic curve over $\mathbb{C}$; the automorphism group, the mapping class group (here we might allow punctures) and if the curve is defined over $\bar{\mathbb{Q}}$ the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ is also acting on the curve. Understanding the above groups is a difficult problem and these actions provide information on both the curve and the group itself. For all the groups mentioned above, the action can often be understood in terms of linear representations, by allowing the group to act on vector spaces and modules related to the curve itself, as the (co)homology groups and section of holomorphic differentials.

For a compact Riemann surface X the automorphism group $\mathrm{Aut}(X)$, consists of all invertible maps $X \to X$ in the category of Riemann surfaces.

A compact Riemann surface minus a finite number of punctures, can be also seen as a connected, orientable topological surface and the mapping class group $\mathrm{Mod}(X)$ can be considered acting on X. The mapping class group is the quotient

$$\mathrm{Mod}(X) = \mathrm{Homeo}^+(X)/\mathrm{Homeo}^0(X),$$

where $\mathrm{Homeo}^+(X)$ is the group of orientation preserving homeomorphisms of X and $\mathrm{Homeo}^0(X)$ is the connected component of the identity in the compact-open topology.

These actions of the above mentioned three types of groups seem totally unrelated and come from different branches of Mathematics. Recent progress in the branch of "Arithmetic topology" provide us with a complete different picture. First the group $\mathrm{Aut}(X)$ can be seen as a subgroup of $\mathrm{Mod}(X)$ consisting of "rigid" automorphisms.

3.1. **Cyclic covers of the projective line.** The authors of [31] focus on curves which are cyclic completely ramified covers of the projective line. These curves form some of the few examples of Riemann surfaces, where explicit computations can be made. The first author considered the automorphism group of these in [27] and also considered the field of moduli versus field of definition in [2], [28].

A ramified cover of the projective curve reduces to a topological cover, when the branch points are removed. By covering map theory these covers correspond to certain subgroups of the fundamental group of the projective line with branch points removed, which is the free group.

Theorem 5. *Consider a set Σ of s points in the projective line and let α be the winding number with respect to these points for a precise definition). Set $X_s = \mathbb{P}^1 - \Sigma$ and fix generators for the fundamental group of X_s, that is $\pi(X_s, x_0) = \langle x_1, \ldots, x_{s-1} \rangle$. In table 2 we give the computation of generators for the open curves involved in this article. The curves on the third column correspond to the quotients of the universal covering space of X_s by the groups of the first column.*

Group	Generators	Curve	Galois group	Homology
F_{s-1}	$x_1, \ldots, x_{s-1}$	$\tilde{X}_s$	F_{s-1}	F_{s-1}/F'_{s-1}
F'_{s-1}	$[x_i, x_j], i \neq j$	Y	F_{s-1}/F'_{s-1}	F'_{s-1}/F''_{s-1}
R_0	$x_1^i x_j x_1^{-i+1}{}_{, 2 \leq j \leq s-1}^{\quad i \in \mathbb{Z}}$	C_s	$\mathbb{Z}$	R_0/R'_0
R_n	$x_1^i x_j x_1^{-i+1}{}_{, 2 \leq j \leq s-1}^{\quad 0 \leq i \leq n-2}$ $x_1^{n-1} x_j, 1 \leq j \leq s-1$	Y_n	$\mathbb{Z}/n\mathbb{Z}$	R_n/R'_n

Table 2. Generators and homology

The homology groups for the cyclic covers C_s (resp. Y_n) can be seen as Galois modules over $\mathbb{Z}$ (resp. $\mathbb{Z}/n\mathbb{Z}$) as follows:

(3.1)
$$H_1(C_s, \mathbb{Z}) = R_0/R'_0 = \mathbb{Z}[\mathbb{Z}]^{s-2} = \mathbb{Z}[t, t^{-1}]^{s-2}$$
$$H_1(Y_n, \mathbb{Z}) = R_n/R'_n = \mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]^{s-2} \bigoplus \mathbb{Z}.$$

The action of the braid group on $H_1(C_s, \mathbb{Z})$ give rise to the Burau representation.

Similar to the above we have that $H_1(C_s, \mathbb{Z}_\ell) = \mathbb{Z}_\ell[\mathbb{Z}]^{s-2}$ but in order to have an action of the absolute Galois group, a larger space is required, namelly the completed group algebra $\mathbb{Z}_\ell[[\mathbb{Z}_\ell]]^{s-2}$. In this way the pro-$\ell$ Burau representation can be defined:

$$\rho_{\text{Burau}} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_{s-2}(\mathbb{Z}_\ell[[\mathbb{Z}_\ell]]).$$

The complete curve $\bar{Y}_n$ has homology

$$H_1(\bar{Y}_n, \mathbb{Z}) = I_{\mathbb{Z}/n\mathbb{Z}}^{s-2},$$

where $I_{\mathbb{Z}/n\mathbb{Z}}$ is the augmentation module of $\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]$.

The later space when tensored with $\mathbb{C}$ gives a decomposition

$$H_1(\bar{Y}_n, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{\nu=1}^{n-1} V_\nu,$$

where each V_ν is the $s-2$-dimensional eigenspace corresponding to eigenvalue $e^{\frac{2\pi i \nu}{n}}$. Each space V_ν gives rise to a representation of the braid group B_s, which is the reduction of the Burau representation at $t \mapsto e^{\frac{2\pi i \nu}{n}}$.

If $n = \ell^k$ then a similar reduction process can be applied to the pro-ℓ Burau representation. We consider the $\ell^k - 1$ non-trivial roots $\zeta_1, \ldots, \zeta_{\ell^k - 1}$ of unity in the algebraically closed field $\bar{\mathbb{Q}}_\ell$. We have

$$\mathbb{Z}_\ell[[\mathbb{Z}_\ell]]^{s-2} \otimes_{\mathbb{Z}_\ell} \bar{\mathbb{Q}}_\ell = \bigoplus_{\nu=1}^{\ell^k - 1} V_\nu,$$

which after reducing $\mathbb{Z}_\ell[[\mathbb{Z}_\ell]] \to \mathbb{Z}_\ell[\mathbb{Z}_\ell/\ell^k\mathbb{Z}_\ell] = \mathbb{Z}_\ell[\mathbb{Z}/\ell^k\mathbb{Z}]$ sending $t \mapsto \zeta_\nu$ gives rise to the representation in V_ν. The modules V_ν in the above decomposition are only $\mathbb{Z}_\ell[[\mathbb{Z}_\ell]]$-modules and $\ker N$-modules, where $N : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}_\ell^*$ is the pro-ℓ cyclotomic character.

Also a pro-ℓ version of the analogue of a Burau representation

$$\rho_{\text{Burau}} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_{s-2}(\mathbb{Z}_\ell[[\mathbf{Z}_\ell]])$$

is given.

We would like to point out that the space $\mathbb{Z}_\ell[[\mathbb{Z}_\ell]]^{s-2}$, contains information of all covers $\bar{Y}_{\ell^k}$ for all $k \in \mathbb{N}$, and equals to the étale homology of a curve $\tilde{Y}$, which appears as a $\mathbb{Z}_\ell$-cover of the projective line, minus the same set of points removed. Going back from the arithmetic to topology we can say that the classical discrete Burau representation can be recovered by all representations of finite cyclic covers $\bar{Y}_n$, since we can define the inverse limit of all mod n representations obtaining the B_s-module $\mathbb{Z}[[\hat{\mathbb{Z}}]]^{s-2}$. This B_s-module in turn contains $\mathbb{Z}[\mathbb{Z}]^{s-2}$ as a dense subset.

Finally an analog of the homology intersection pairing can be interpreted as an intersection pairing using the Galois action on the Weil pairing for the Tate module. For a free $\mathbb{Z}$ (resp. $\mathbb{Z}_\ell$)-module of rank $2g$, endowed with a symplectic pairing $\langle \cdot, \cdot \rangle$ the symplectic group is defined as

$$\mathrm{Sp}(2g, \mathbb{Z}) = \{M \in \mathrm{GL}(2g, \mathbb{Z}) : \langle Mv_1, Mv_2 \rangle = \langle v_1, v_2 \rangle\}$$

and the generalized symplectic group is defined as

$$\mathrm{GSp}(2g,\mathbb{Z}) = \{M \in \mathrm{GL}(2g,\mathbb{Z}_\ell) : \langle Mv_1, Mv_2\rangle = m\langle v_1, v_2\rangle, \text{ for some } m \in \mathbb{Z}_\ell^*\}.$$

In the topological setting the pairing is the intersection pairing and we have the following representation

$$\rho : B_{s-1} \to \mathrm{Sp}(2g,\mathbb{Z})$$

We employ properties of the Weil pairing in order to show that we have a representation

$$\rho' : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GSp}(2g,\mathbb{Z}_\ell)$$

as an arithmetic analog of the braid representation ρ.

The assumption of complete ramification is removed in [26] where the most general cyclic cover of the projective line is considered. This article contains a comparison with the dual space of holomorphic differentials using the Hodge decomposition and the classical results of Ellingsrud-Lønsted [14].

3.2. **Fermat covers.** In [30] the case of (generalized) Fermat covers of the projective line is studied. A generalized Fermat curve $C_{k,s-1}$ is a non-singular projective curve with an automorphism group $H_0 \cong (\mathbb{Z}/k\mathbb{Z})^{s-1}$. The fundamental groups of both the open and the completed curves are computed used combinatorial group theory techniques. he pro-l homology $H_1(C_{l^k,s-1},\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_l$ is interpreted via the pro-l completion of the fundamental groups11:

$$\mathrm{rank}_{\mathbb{Z}_l} H_1(C_{l^k,s-1},\mathbb{Z}_l) = (s-1)(l^k)^{s-1} + 2 - s(l^k)^{s-2}$$

12The homology decomposes into characters χ_i of H_0:

$$H_1(C_{l^k,s-1},\mathbb{F}) = \bigoplus_i \mathbb{F}\cdot C(i)\chi_i,$$

where $C(i)$ is a multiplicity constant determined by the number of zero-indices in i. o study the action of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, the authors utilize Alexander modules and the Crowell exact sequence:

$$0 \to R/R' \xrightarrow{\theta_1} \mathcal{A}_\psi \xrightarrow{\theta_2} \mathcal{J} \xrightarrow{\epsilon} \mathbb{Z}_l \to 0$$

The Alexander module $\mathcal{A}_\psi$ is computed as the cokernel of a matrix Q of Fox derivatives. For generalized Fermat curves, the matrix Q involves the sums $\Sigma_i = 1 + \bar{x}_i + \cdots + \bar{x}_i^{l^k-1}$.

The article connects these geometric structures to Ihara's theory of pro-l braid groups and the Magnus embedding.

- Ihara Representation: The absolute Galois group acts on the pro-l free group $\mathfrak{F}_{s-1}$ via $Ih_S : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to Aut(\mathfrak{F}_{s-1})$21.
- Magnus Embedding: This map sends $\mathfrak{F}_{s-1}$ into a non-commutative power series algebra $\mathbb{Z}_l[[u_1,\dots,u_{s-1}]]_{nc}$ via $x_i \mapsto 1 + u_i$.
- Gassner and Burau Representations: The authors demonstrate how to transition from the Gassner representation (associated with generalized Fermat curves) to the Burau representation (associated with cyclic covers) by reducing variables.

3.3. **Heisenberg curves.** In [29] the Heisenberg curves are studied. These are covers of the projective line minus three points ($P^1 \setminus \{0,1,\infty\}$) by the non-abelian Heisenberg group modulo n (H_n). They also appear as unramified cover of the Fermat curve if n is odd, but it becomes a ramified cover if n is even.

A major technical result is the explicit computation of the fundamental group (R_{Heis_n}) of the curve. Schreier's lemma is used to find a set of $n^3 + 1$ free generators for this group. The generators are described in terms of the "Galois action", showing how the Heisenberg group H_n acts on the abelianization of this fundamental group via conjugation.

The action of Artin's braid group B_3 on the curve is described. Two cases are distinguished:

- Odd n: The Heisenberg curve's fundamental group is "characteristic", meaning it stays invariant under the braid group action.
- Even n: A significant discovery is that for even n, the braid group can map the Heisenberg curve to a non-isomorphic curve. This provides an interesting case study in the "field of moduli versus field of definition", proving that the Heisenberg curve and its "braid-conjugate" are not isomorphic over $\mathbb{Q}$.

Finally, a complete description of the curve's homology ($H_1(X_H,\mathbb{F})$) over a field of characteristic zero is given. This homology is decomposed into irreducible representations of the Heisenberg group and an explicit formula for the coefficients h_{ijs} that determine how many times each irreducible character χ_{ijs} appears in the homology group is given.

3.4. **Tanakian Approach to $\mathbb{F}_1$.** M. Kapranov and A. Smirnov in [24] defined a vector space V over $\mathbb{F}_1$ to be just a set, then $\dim_{\mathbb{F}_1} V = \#V$, and then the general linear group of V is just the symmetric group, $\mathrm{GL}(V) = S_{\#V}$. They also defined algebraic extensions of $\mathbb{F}_{1^n}$ over $\mathbb{F}_1$ to be $\{0\} \cup \mu_n$, where μ_n are the n-th roots of unity. Most interesting for our purposes is the definition of the general linear group of the polynomial ring over $\mathbb{F}_1$ to be the full braid group, i.e. $\mathrm{GL}_d(\mathbb{F}_1[t]) = B_d$, while $\mathrm{GL}_d(\mathbb{F}_{1^n}[t])$ is $(\mathbb{Z}/n\mathbb{Z})^d \rtimes B_d$, and $\mathrm{GL}_d(\mathbb{F}_{1^n}) \cong (\mathbb{Z}/n\mathbb{Z})^d \rtimes S_d$, the later group is the complex reflection group $G(n, 1, r)$. The complex reflection groups were classified by G. Shephard and J. Todd, [41] and their representation theory and Hecke algebra structure is an active research topic, see [6]. Also the group $\mathrm{GL}_d(\mathbb{F}_{1^n}[t])$ is essential to the study of framed knots and links, see [18] where the P.I. also contributed.

Kapranov and Smirnov explained their definition, realizing the braid group as the fundamental group of the space of polynomials without multiple roots, i.e. as the the fundamental group of the configuration space of the roots, while the group $\mathrm{GL}_d(\mathbb{F}_q)$ acts on

$$\bar{\mathbb{F}}_q^d - \bigcup_{(\alpha_1,\ldots,\alpha_d)\in\mathbb{F}_q^s-\{0\}} \left\{(x_1,\ldots,x_d) : \sum \alpha_i x_i = 0\right\},$$

i.e. on d-tuples which are not $\mathbb{F}_q$-linear depended. By a fundamental theorem on additive polynomials, see [17, th. 1.2.1], the quotient of this action is just the space of q-additive polynomials

$$x^{q^d} + a_1 x^{q^{d-1}} + \cdots + a_{d-1}x^q + a_d x, a_d \neq 0.$$

Drinfeld, identified the space of elliptic $\mathbb{F}_q[t]$-modules of rank d with level N-structure, for every $N \in \mathbb{F}_q[t]$, as an unramified covering of the space of additive polynomials with Galois group $\mathrm{GL}_d(\mathbb{F}_q[t]/N)$. This allows us to see the profinite completion of $\mathrm{GL}_d(\mathbb{F}_q)$ embedded in the fundamental group of the space of q-polynomials.

In order to pursue the analogy further and relate it to Ihara's theory one needs to see the action of the profinite completion of $\mathrm{GL}_q(\mathbb{F}_q)$ as an action of the group to the fiber. Since the fiber consists of vector spaces, we believe that a Tannakian approach through local systems can be applied here, see [42, sec. 2.6, chap. 6]. Indeed geometrically the action of the Braid group $\mathrm{GL}_d(\mathbb{F}_1[t])$ on the free group, can be interpreted as the action of the fundamental group of the base, i.e. the configuration space of the roots, to the fundamental group of the fiber i.e. the disc with d-points removed, see [23]. In the $\mathrm{GL}_d(\mathbb{F}_q[t])$ case the fundamental group of the base has to act on fibers that are vector spaces and it acts as linear group automorphisms, as the Tannakian approach dictates.

4. Topological aspects of field of moduli/definition

In [25] we pursue this analogy further by transferring classical arithmetic notions to the topological setting. A central object of study in the arithmetic of curves is the "field of moduli/definition problem". The primary objective of [25] is to bridge the two perspectives by formulating the arithmetic setup of field descent in a purely group-theoretic language, effectively defining the analogous theory for surfaces. The approach is guided by the principle that **fields descend as groups ascend**.

By interpreting the choice of a field extension as the choice of a subgroup of the absolute Galois group—and, by analogy, a subgroup of the Mapping Class Group—it is shown that many deep results in algebraic geometry regarding group extensions and equivariant structures have natural topological incarnations. Specifically, it is proposed that the arithmetic homotopy exact sequence [21]:

$$1 \to \Pi_{K_s}(B^*) \to \Pi_K(B^*) \to \mathrm{Gal}(K_s/K) \to 1, \tag{4.1}$$

where B^* represents the punctured base variety, is analogous to the Birman exact sequence [15] in topology:

$$1 \to \pi_1(B^*, b_0) \to \mathrm{Mod}(B^*, b_0) \to \mathrm{Mod}(B^*) \to 1. \tag{4.2}$$

For instance, it is well known the existence of a K-rational point yields the splitting of (4.1). It is demonstrated that this splitting can also be established via a group-theoretic framework that applies with equal rigor to (4.2). Furthermore, notice that the splitting of these sequences is generally obstructed in both settings. On the one hand, the condition that rational points exist is often restrictive; on the other, the Birman exact sequence is known not to split for closed surfaces.

Main Results & Ideas Employed. We now outline the primary contributions of [25], organized by the central themes that bridge the arithmetic and topological perspectives.

The Arithmetic-Topology Viewpoint. The first theme concerns the structural identity between the arithmetic, geometric and topological fundamental groups. Grothendieck's theory of the fundamental group provides the arithmetic short exact sequence:

$$1 \to \Pi_{K_s}(B^*) \to \Pi_K(B^*) \to \mathrm{Gal}(K_s/K) \to 1.$$

which yields the continuous homomorphism of profinite groups

$$\mathrm{Gal}(K_s/K) \to \mathrm{Out}(\Pi_{K_s}(B^*)).$$

Similarly, in the topological setting, the Dehn-Nielsen-Baer theorem (Theorem **??**) in conjunction with the Birman exact sequence yield a homomorphism

$$\mathrm{Mod}(B^*) \to \mathrm{Out}(\pi_1(B^*)).$$

As mentioned previously, the existence of a $\mathrm{Gal}(K_s/K)$-fixed point leads to the splitting of the arithmetic sequence (4.1). We can apply this principle to the topological setting using purely group-theoretic language:

Proposition A. . *The short exact sequence*

$$1 \to \pi_1(B^*, b_0) \to \Pi_{\mathrm{Mod}(B^*)} \to \mathrm{Mod}(B^*) \to 1.$$

splits if there is a point P such that all $\sigma \in \mathrm{Mod}(B^*)$ have a homeomorphism representative $\hat{\sigma}$ that fixes P.

We observe that a significant number of proofs regarding sequences (4.1), (4.2) and their associated outer automorphisms can be formulated in a unified way that applies to both settings simultaneously. The power of this unified language extends far beyond the splitting of exact sequences. By abstracting the notion of "field" into "group action", we recover several deep geometric properties purely through group theory:

(1) **Regularity:** The arithmetic condition that a field extension $K(X)/K(B)$ contains no new constants is classically verified by intersecting with the separable closure K_s of K. It is shown that this is equivalent to the purely group-theoretic condition of index preservation: $[\Pi_A : H] = [\Pi_1 : R]$. This allows us to define and study the notion of "regularity" of covers in topology, where no base field exists.

(2) **Monodromy Actions:** Through the theory of Dèbes et al. [10, 12, 11], the field extensions $K(X)/K(B)$ correspond to monodromy representations of the arithmetic fundamental group $\Pi_K(B^*)$. We demonstrate that the same correspondence applies to the Birman exact sequence and covers $X \to B$ "defined" over $A \subseteq \mathrm{Mod}(B^*)$ correspond to monodromy representations of Π_A. In both settings, the groups $\mathrm{Gal}(K_s/K)$ and $\mathrm{Mod}(B^*)$ act on the respective representations. This allows us to treat arithmetic and topological symmetries on equal footing.

(3) **Cohomological Obstructions:** Finally, the obstruction of descending an arithmetic cover to its field of moduli-also related to the splitting of (4.1)-is identified as a specific Galois cohomology class, originally by Dèbes and Douai [10]. This group cohomological description applies verbatim to both the arithmetic descent of varieties and the topological group ascent of covers via the idea described next.

Fields Descend as Groups Ascend. This simple idea follows from the following observation. Given a variety X defined over a field K and a field extension L/K, then one can extend it to a variety $X_L := X \times_K L$. One could also extend it similarly to a variety X_{K_s}, where K_s denotes a separable closure of K that contains L. Then we know that X_{K_s} is invariant under the action of the absolute Galois group $\mathrm{Gal}(K_s/K)$. However, if we view X_{K_s} as arising from X_L, it is effectively invariant under the subgroup $\mathrm{Gal}(K_s/L)$. Thus, the smaller the field of definition (i.e., descending from L to K), the larger the group of invariance (i.e., ascending from $\mathrm{Gal}(K_s/L)$ to $\mathrm{Gal}(K_s/K)$).

This inverse relationship allows us to translate the arithmetic problem of finding a minimal field of definition into the group-theoretic problem of finding a maximal subgroup of invariance (or definition) within a larger ambient group. We apply this philosophy to the topological setting by replacing the Galois tower with subgroups of the Mapping Class Group $\mathrm{Mod}(B^*)$.

Fields & Groups of Moduli, Definition & Invariance. The notion of symmetry is refined by distinguishing between three levels of group actions, applicable to both arithmetic and topology. For an object X and a subgroup A of either $\mathrm{Mod}(B^*)$ or $\mathrm{Gal}(K_s/K)$:

(1) The *Group of Moduli* A_X^{mod} consists of elements $\sigma \in A$ such that ${}^{\sigma}X$ is isomorphic to X. In the arithmetic setting, its fixed field is the *field of moduli*.

(2) The *Group of Definition* A_X^{def} is the subgroup where these isomorphisms satisfy the Weil cocycle condition (or equivalently, admit the structure of a linearization of X). This corresponds to the *field of definition*.

(3) The *Group of Invariance* A_X^{inv} consists of the elements preserving strictly X. This corresponds to the *field of invariance*.

The subtle distinction between the field of moduli and the field of definition is discussed. While they might coincide in most cases, for example for elliptic curves, we discuss a specific example, where the field of moduli is strictly smaller than the field of definition, a phenomenon detected by the obstruction of the Weil cocycle in the group of definition.

Weil's Topological Ascent and Definability. In the topological realm, the absence of an underlying field structure necessitates a purely group-theoretic reconstruction of the notion of "field of definition." We address this by replacing field extensions with subgroups of the Mapping Class Group $\mathrm{Mod}(B^*)$.

We introduce the notation $X \to (B, A)$ to denote a cover defined over a subgroup $A \leq \mathrm{Mod}(B^*)$. This definition captures the topological essence of a curve being defined over a subfield: the geometric object remains invariant under the symmetries corresponding to that subfield.

A central result of this paper is the formulation of a topological analogue to Weil's descent theorem. In the arithmetic setting, descending a variety from a field L to a subfield L_0 requires a **Weil descent datum**: a family of isomorphisms $\{f_\sigma \colon X \xrightarrow{\simeq} {}^\sigma X\}_{\sigma\in\mathrm{Gal}(L/L_0)}$ satisfying the cocycle condition

$$f_{\sigma\tau} = {}^\sigma f_\tau \circ f_\sigma.$$

In our topological framework, this transforms into an *ascent* problem: extending the definition of a cover from a group A to a larger group A'. We define a **Mapping Class Group ascent datum** as a family of pairs of homeomorphisms $\{(f_\sigma, \hat{\sigma})\}_{\sigma\in A'/A}$, where $f_\sigma \colon X \to {}^\sigma X$ maps the cover to its twist, satisfying the analogous cocycle condition:

$$(f_{\sigma\tau}, \widehat{\sigma\tau}) = ({}^\sigma f_\tau, {}^\sigma\hat{\tau}) \circ (f_\sigma, \hat{\sigma}).$$

The comparison reveals a striking duality: the arithmetic descent datum allows a geometric object to descend to a smaller field, while the topological ascent datum allows the object to ascend to a larger group of symmetries.

Theorem A. *Let $A \lhd_{\mathrm{f}} A' \leq \mathrm{Mod}(B^*)$ and let $X \to (B, A)$ be a cover defined over A. The cover is definable over the larger group A' if and only if it admits a Mapping Class Group ascent datum with respect to the quotient A'/A.*

This result, together with the topological formulation of Dèbes, completes our arithmetic-topology viewpoint, establishing that the cohomological obstructions to enlarging the group of definition in topology are formally identical to the obstructions of descending the field of definition in arithmetic.

Descent on Monodromy. In the arithmetic setting, the descent theory of covers is governed by the results of Dèbes and Douai [10]. Their *Main Theorem* establishes that for a cover $X \to B$ defined over K_s with field of moduli K, the obstruction to K being a field of definition lies in a specific cohomology class in $H^2(K, Z(G))$, where G is the automorphism group of the Galois closure. We adapt this result to the topological setting utilizing purely group-theoretic techniques. A crucial step is the group-theoretic characterization of the "extension of constants in the Galois closure" which plays a central role in the obstruction. By replacing the Galois action of $\mathrm{Gal}(K_s/K)$ with the action of a subgroup $A \leq \mathrm{Mod}(B^*)$, we apply the same techniques to the monodromy representations $\Pi_A \to G$ to derive the topological analogues.

Let $f \colon X \to B$ be cover defined over $A \leq \mathrm{Mod}(B^*)$ with group of moduli A' such that A'/A is a finite group, and let G be the associated automorphism group of the Galois closure. Let N, C denote the normalizer and centralizer of G in S_d, respectively, where d is the degree of the cover. The first obstruction, which is non-trivial only for non-Galois covers, relates the extension of constants in the Galois closure to a homomorphism $\Lambda \colon A'/A \to N/G$.

Theorem B. *Mapping class group formulation of* [10, *Main Theorem* (I)]

Let $f \colon X \to (B, A)$ be a (G)-cover with group of moduli A' and $\overline{\phi} \colon \Pi_{A'} \to N/C$ associated to the monodromy representation $\Pi_A \to N$ of f. Then there exists a unique homomorphism $\lambda \colon A'/A \to N/CG$ compatible with $\overline{\phi}$. Each A'-model of f yields a lift of λ to $\Lambda \colon A'/A \to N/G$. Thus, the existence of a homomorphism Λ extending λ is necessary for the cover to be defined over its group of moduli.

Intuitively, this condition requires that the extension to the Galois closure is compatible with the G-action. The subsequent theorem, which identifies the *Main Obstruction*, determines whether such a lift $\Lambda \colon A'/A \to N/G$ arises from an A'-model of the cover.

Theorem C. *Mapping class group formulation of* [10, *Main Theorem* (II)]
Let A' be the group of moduli of the (G)-cover of $f\colon X \to (B, A)$ relative to A'/A. Assume that the necessary condition of Theorem B is satisfied and fix a lift $\Lambda\colon A'/A \to N/G$ of λ. The group of moduli A' is a group of definition for f if and only if a specific 2-cocycle Ω_Λ has its inverse in the image of the coboundary operator $\delta\colon H^1(A'/A, CG/G) \to H^2(A'/A, Z(G))$. That is, there is a 1-cocycle θ such that $\delta(\theta)\Omega_\Lambda$ is trivial in $H^2(A'/A, Z(G))$.

The next result provides a simplified criterion for the case where the exact sequence $1 \to \Pi_A \to \Pi_{A'} \to A'/A \to 1$ splits. In particular, the A-model $f\colon X \to B$ with monodromy representation $\Psi\colon \Pi_A \to N$ can ascend to the group of moduli A' if and only if there exists an action of A'/A on N compatible with Ψ, meaning it respects the semi-direct product structure $\Pi_{A'} \cong \Pi_A \rtimes A'/A$.

Theorem D. *Mapping class group formulation of* [10, *Main Theorem* (III)]
Retain the assumptions of Theorem B and assume further that $s\colon A'/A \to \Pi_{A'}$ is a section. The cover is defined over its group of moduli A' if and only if $\overline{\phi} \circ s\colon A'/A \to N/C$ admits a lift $\phi : A'/A \to N$.

Birman-Hilden Property. The relationship between the mapping class group of a cover and that of the base is studied. The classical *Birman-Hilden property* asks when mapping classes of the covering space surject to mapping classes of the base space. In our framework, the choice of a cover $X \to (B, A)$ defined over $A \le \mathrm{Mod}(B^*)$ induces a homomorphism $A \to \mathrm{Out}(\pi_1(X))$. We view this as a dual property to the Birman-Hilden one. While the well-known theory of the Birman-Hilden property focuses on compatibility of isotopy representatives when mapping $\mathrm{Mod}(X)$ to $\mathrm{Mod}(B^*)$, we provide a classical reinterpretation in group theoretic terms to accompany the rest of the article. Furthermore, we explore the connection between this property and the action of $\mathrm{Mod}(B^*)$ on the Teichmüller space $T(X)$ of X as a genus g surface.

Equivariant Categories and Weil's Theorem. The framework of equivariant categories provides the natural setting for our unification of descent theory. A core theme of this work is the realization that Weil's classical descent condition—historically formulated via cocycles $\{f_{\sigma\tau} = {}^{\sigma}f_\tau \circ f_\sigma\}$—is precisely the condition for an object to admit a *linearization* in an equivariant category. By identifying either Weil's descent data or its topological counterpart, mapping class group ascent data, with linearizations, we obtain the following categorical equivalences for both arithmetic varieties and topological covers.

We denote by Var_L the category of L-varieties and by $\mathsf{Cov}_{(B,A)}$ the category of Galois covers $X \to (B, A)$ with branch divisor in B and defined over A.

Theorem E. *Let L/L_0 a finite Galois field extension and $A \lhd_{\mathrm{f}} A'$ be subgroups of $\mathrm{Mod}(B^*)$. The category of varieties Var_L is equipped with a natural strict Galois L/L_0 action and the category of covers $\mathsf{Cov}_{(B,A)}$ is equipped with a natural non-strict action of the mapping class group quotient A'/A. An object X of either of the categories is definable over the base (L_0 or A', respectively) if and only if it admits a linearization with respect to the natural group action on the category. In particular, we obtain equivalences of categories:*

$$\mathsf{Var}_L^{\mathrm{Gal}(L/L_0)} \simeq \mathsf{Var}_{L_0} \quad \text{and} \quad \mathsf{Cov}_{(B,A)}^{A'/A} \simeq \mathsf{Cov}_{(B,A')}.$$

This categorical perspective clarifies the definitions provides a different viewpoint to the group of definition. Note that both parts of this theorem are induced by Weil's arithmetic and topological theorems respectively.

A central theme of this work is the reinterpretation of *descent theory*. Historically, the problem of determining the field of definition of an algebraic variety was solved by A. Weil in his seminal 1956 work [43]. Weil introduced the concept of *descent data*—a family of isomorphisms satisfying a specific cocycle condition—which allows one to "descend" a variety defined over a field L to a smaller subfield L_0. While Weil's theorem is a cornerstone of arithmetic geometry, its topological counterpart—descending a cover to a larger group—has remained less formalized in the language of descent data. It seems that the Weil cocycle condition, the standard tool for the descent of varieties, can be more naturally expressed as a linearization in the language of *equivariant categories*.

Equivariant categories arise naturally in the study of sheaves on orbit spaces. This concept traces back to Grothendieck's seminal Tôhoku paper [20], where he investigated the relationship between G-equivariant sheaves on a topological space X and sheaves on the orbit space X/G. This framework was formalized by Mumford and Fogarty in their Geometric Invariant Theory [37], establishing the definitive notion of a G-equivariant sheaf. The theory has since flourished; notably, Bernstein and Lunts [5] pioneered the definition of equivariant bounded derived categories for G-spaces, a framework later extended to the scheme-theoretic setting by Achar [1]. Consequently, the study of group actions has been generalized to arbitrary categories, giving rise to abstract equivariant categories. These categories are now viewed as non-commutative categorical quotients, forming a distinct and active area of research. For instance, Elagin [13] established criteria for the equivariant category of a triangulated category to inherit a canonical triangulated structure, while Beckmann and Oberdieck [3] determined the equivariant categories of elliptic curves with respect to so-called Calabi–Yau group actions.

A classical algebraic prototype is provided by a finite group acting on a ring R: the equivariant category $(\mathrm{Mod}{-}R)^G$ is equivalent to the category of modules over the skew group algebra $R * G$, a correspondence extensively studied by Reiten and Riedtmann [40]. In the geometric setting, it is well known that a G-action on a scheme X induces an action on $\mathsf{Coh}(X)$ such that the equivariant category $\mathsf{Coh}^G(X)$ is equivalent to the category of coherent sheaves on the quotient stack $[X/G]$.

This philosophy is pushed further by extending the descent formalism to the *bounded derived category* $\mathsf{D}^{\mathsf{b}}(X)$. Rather than viewing the derived category merely as a homological invariant, it is treated as a geometric object in its own right, subject to Galois actions and descent. A canonical group action on $\mathsf{D}^{\mathsf{b}}(X)$ is constructed, arising solely from a Weil descent datum on X, thereby providing a categorical proof of Weil's theorem that naturally encompasses derived invariants.

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