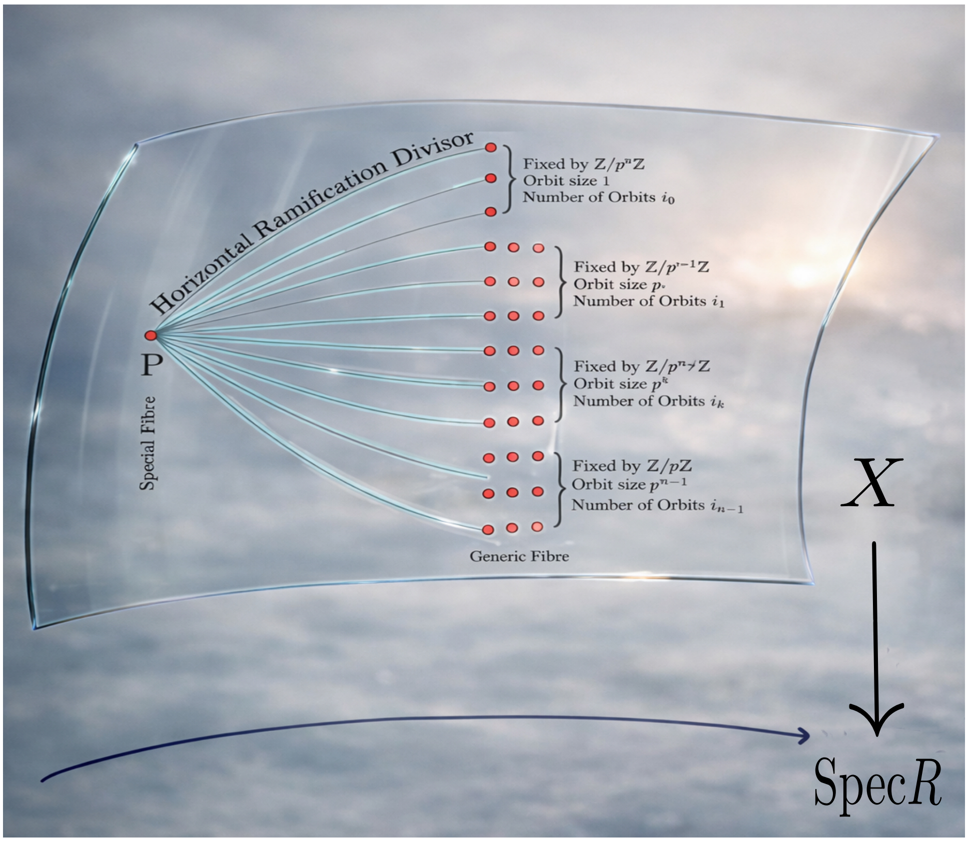
Group Actions on Algebraic Curves
GAAC

Deformation Theory and Liftings



Research Team:

Karagiannis	**Kostas**
Karakikes	**Miltiadis**
Karanikolopoulos	**Sotiris**
Kontogeorgis	**Aristides**
Lygdas	**Orestis**
Manousoy Sotiropoulou	**Kostantia**
Noulas	**Dimitrios**
Stergiopoulou	**Dimitra-Dionysia**
Terezakis	**Alexis**

The research project is implemented in the framework of H.F.R.I Call "Basic research Financing (Horizontal support of all Sciences)" under the National Recovery and Resilience Plan "Greece 2.0" funded by the European Union Next Generation EU(H.F.R.I. Project Number: 14907.

Contents

1. Introduction

In some respects, deformation theory is as old as algebraic geometry itself. This historical rootedness stems from the fact that any algebro-geometric object can be "deformed" by simply varying the coefficients of its defining equations. While classical geometers were certainly aware of this phenomenon, achieving a rigorous understanding of what it means to "deform" an object leads into some of the most technically challenging areas of the discipline.

More specifically, deformation theory serves as a formalization of the Kodaira, Nirenberg, Spencer, and Kuranishi (KNSK) approach. The analytic deformation theory of complex structures was developed by Kodaira and Spencer [33, 34] and completed by Kuranishi [45], using elliptic methods introduced by Nirenberg [54].

The fundamental ideas of this theory are clearly outlined in the series of Bourbaki seminar exposés by Grothendieck, collectively known as "Fondements de la Géométrie Algébrique", [20] which formalized and translated the KNSK approach into the language of schemes.

1.1. The Kodaira-Nirenberg-Kuranishi Approach to Deformation Theory.

The deformation theory of complex manifolds and complex structures was profoundly shaped by the work of Kodaira, Spencer, Nirenberg, and Kuranishi in the 1950s and 1960s. Their analytic approach provides a local description of the moduli space of complex structures on a fixed differentiable manifold, revealing it as the zero locus of a nonlinear analytic map between finite-dimensional complex vector spaces.

Let X be a compact complex manifold, and let M denote the underlying smooth manifold. A deformation of the complex structure on X is a smooth family of complex structures on M, parametrized by a germ of a complex space $(S, 0)$, such that the fiber over 0 is biholomorphic to X. The central goal of deformation theory is to describe such deformations up to biholomorphism and to understand the local structure of the corresponding moduli space.

The Kodaira-Nirenberg-Kuranishi theory begins by expressing complex structures in analytic terms. A complex structure on M is determined by a decomposition of the complexified tangent bundle

$$T_M \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0} \oplus T^{0,1},$$

or equivalently by a $\bar{\partial}$-operator acting on smooth functions. Small deformations of the complex structure are described by perturbations of this operator, which may be identified with $(0,1)$-forms with values in the holomorphic tangent bundle,

$$\phi \in A^{0,1}(X, T_X).$$

The integrability of a perturbed almost complex structure is governed by the Maurer-Cartan equation

$$\bar{\partial}\phi + \tfrac{1}{2}[\phi, \phi] = 0,$$

where the bracket is induced by the Lie bracket of vector fields. This equation encodes the vanishing of the Nijenhuis tensor and ensures that the deformed structure defines a genuine complex manifold.

A fundamental result of Kodaira and Spencer identifies the space of infinitesimal deformations of X with the Dolbeault cohomology group

$$H^1(X, T_X),$$

while obstructions to extending infinitesimal deformations to higher order lie in

$$H^2(X, T_X).$$

Thus, the deformation problem acquires a cohomological interpretation, with explicit tangent and obstruction spaces.

The analytic core of the theory is due to Nirenberg and Kuranishi. Using elliptic operator theory and the implicit function theorem, they showed that the infinite-dimensional Maurer-Cartan equation can be reduced to a finite-dimensional problem. After fixing a Hermitian metric on X and imposing a suitable gauge condition, one constructs a finite-dimensional complex analytic space, called the *Kuranishi space*, which parametrizes small deformations of the complex structure on X.

More precisely, there exists a holomorphic map, the *Kuranishi map*,

$$\kappa\colon H^1(X, T_X) \longrightarrow H^2(X, T_X),$$

such that the Kuranishi space is locally isomorphic to the zero locus $\kappa^{-1}(0)$. This space enjoys a universal property: any sufficiently small deformation of X is induced by a unique morphism from its base space to the Kuranishi space.

In particular, when the obstruction space $H^2(X, T_X)$ vanishes, the Kuranishi space is smooth, and its dimension equals $\dim H^1(X, T_X)$. In this unobstructed case, the Kuranishi space provides a local analytic model for the moduli space of complex structures near X. Especially, when X is a nonsingular curve then the 2-cohomology $H^2(X, T_X)$ vanishes.

The Kodaira-Nirenberg-Kuranishi approach thus furnishes a powerful analytic framework for deformation theory, synthesizing techniques from complex analysis, differential geometry, and cohomology. It has served as a foundational paradigm for numerous later developments, including the deformation theory of vector bundles, complex analytic spaces, and algebraic varieties, and continues to play a central role in the study of moduli problems in geometry.

2. Schlessinger's Approach to Deformation Theory

Schlessinger's approach to deformation theory provides an abstract and functorial framework for studying infinitesimal and formal deformations of algebraic and geometric objects. Introduced in [65], this theory axiomatizes deformation problems in terms of covariant functors on categories of Artinian local rings and establishes precise criteria for the existence of versal and pro-representable deformation spaces.

2.1. **The category of Artin rings.** Let k be a field and let $\mathcal{C}$ denote the category whose objects are local Artinian k-algebras Γ with residue field k, and whose morphisms are local k-algebra homomorphisms inducing the identity on the residue field. We write $\mathfrak{m}_\Gamma$ for the maximal ideal of Γ.

A *small extension* in $\mathcal{C}$ is a short exact sequence

$$0 \longrightarrow I \longrightarrow \Gamma' \longrightarrow \Gamma \longrightarrow 0 \tag{2.1}$$

such that $I \cdot \mathfrak{m}_{\Gamma'} = 0$. In this case, I is naturally a finite-dimensional k-vector space.

2.2. **Deformation functors.** A *deformation functor* is a covariant functor

$$D : \mathcal{C} \longrightarrow \mathrm{Sets}$$

satisfying $D(k)$ is a singleton. For each $\Gamma \in \mathcal{C}$, the set $D(\Gamma)$ parametrizes equivalence classes of deformations of a fixed object over $\mathrm{Spec}(\Gamma)$.

The functoriality of D encodes base change for deformations. Given a morphism $\Gamma' \to \Gamma$ in $\mathcal{C}$, the induced map $D(\Gamma') \to D(\Gamma)$ corresponds to restriction of deformations.

2.3. **Infinitesimal deformations.** The vector space

$$t_D := D\big(k[\epsilon]/(\epsilon^2)\big)$$

is called the *tangent space* of the deformation functor D. It represents first-order infinitesimal deformations. In geometric examples, this space often admits a cohomological description, for instance as $H^1(X, T_X)$ for deformations of a smooth variety X.

2.4. **Schlessinger's conditions.** Schlessinger introduced four conditions, denoted (H_1)-(H_4), which control the behavior of a deformation functor with respect to fiber products in $\mathcal{C}$.

Let $\Gamma', \Gamma'' \to \Gamma$ be morphisms in $\mathcal{C}$. There is a natural map

$$D(\Gamma' \times_\Gamma \Gamma'') \longrightarrow D(\Gamma') \times_{D(\Gamma)} D(\Gamma'').$$

- (H_1) This map is surjective when $\Gamma'' \to \Gamma$ is a small extension.
- (H_2) The map is bijective when $\Gamma = k$ and $\Gamma'' = k[\epsilon]/(\epsilon^2)$.
- (H_3) The tangent space t_D is finite dimensional over k.

- (H_4) The map is bijective for all small extensions $\Gamma'' \to \Gamma$.

These conditions encode the existence and uniqueness properties of liftings of deformations across small extensions.

2.5. **Pro-representability and versal deformations.** A deformation functor D is said to be *pro-representable* if there exists a complete local Noetherian k-algebra R with residue field k such that

$$D(\Gamma) \cong \mathrm{Hom}_{\mathrm{cont}}(R, \Gamma)$$

for all $\Gamma \in \mathcal{C}$.

Schlessinger proved that D is pro-representable if and only if it satisfies conditions (H_1)-(H_4). If only (H_1)-(H_3) hold, then D admits a *versal deformation*, represented by a formally smooth morphism from a pro-representable functor.

2.6. **Obstructions.** Given a small extension as in (2.1) and an element $x \in D(\Gamma)$, the problem of lifting x to an element of $D(\Gamma')$ is governed by an obstruction lying in a finite-dimensional k-vector space, called the *obstruction space*. In many geometric situations, this space can be identified with a second cohomology group, such as $H^2(X, T_X)$.

Schlessinger's framework thus isolates the formal properties of deformation theory and provides a unified language encompassing deformations of schemes, complex structures, representations, and many other algebraic objects.

3. Deformation Theory of curves with automorphismns

The deformation theory of curves with automorphisms serves as a significant generalization of classical curve deformation theory. It is intimately linked to the lifting problem, as liftings from characteristic $p > 0$ to characteristic zero can be framed in terms of sequences of local Artin rings.

Following Schlessinger's framework, Bertin and Mézard [4] introduced the global deformation functor D_{gl}, analyzing it through the lens of Grothendieck's equivariant cohomology theory [19]. Within Schlessinger's approach, the primary objectives are to identify the tangent space of the deformation functor $D_{\mathrm{gl}}(k[\epsilon])$ and to determine the potential obstructions to lifting a deformation over a local Artin ring Γ to a small extension $\Gamma' \to \Gamma$.

Let Λ be a complete local Noetherian ring with residue field k, where k is an algebraically closed field of characteristic $p \geq 0$. Let $\mathcal{C}$ denote the category of local Artin Λ-algebras with residue field k, whose morphisms are local Λ-algebra homomorphisms

$$\phi \colon \Gamma' \to \Gamma$$

satisfying $\phi^{-1}(\mathfrak{m}_\Gamma) = \mathfrak{m}_{\Gamma'}$.

The deformation functor of curves with automorphisms is a functor

$$D_{\mathrm{gl}} \colon \mathcal{C} \longrightarrow \mathrm{Sets}, \qquad \Gamma \longmapsto \left\{ \begin{array}{l} \text{Equivalence classes} \\ \text{of deformations of} \\ \text{pairs } (X,G) \text{ over } \Gamma \end{array} \right\},$$

defined as follows.

Let G be a subgroup of the automorphism group $\mathrm{Aut}(X)$. A deformation of the pair (X, G) over a local Artin ring Γ consists of a proper and smooth family of curves

$$X_\Gamma \longrightarrow \mathrm{Spec}(\Gamma),$$

together with a group homomorphism

$$G \longrightarrow \mathrm{Aut}_\Gamma(X_\Gamma),$$

such that there exists a G-equivariant isomorphism

$$\phi \colon X_\Gamma \otimes_{\mathrm{Spec}(\Gamma)} \mathrm{Spec}(k) \longrightarrow X$$

between the special fibre of the family and the original curve X.

Two deformations X_Γ^1 and X_Γ^2 of (X, G) over Γ are said to be equivalent if there exists a G-equivariant isomorphism

$$\psi \colon X_\Gamma^1 \longrightarrow X_\Gamma^2$$

which restricts to the identity on the special fibre and makes the following diagram commute:

$$\begin{array}{ccc} X^1_\Gamma & \xrightarrow{\psi} & X^2_\Gamma \\ & \searrow \quad \swarrow & \\ & \mathrm{Spec}(\Gamma) & \end{array}$$

Given a small extension of local Artin rings

$$0 \longrightarrow E \cdot k \longrightarrow \Gamma' \longrightarrow \Gamma \longrightarrow 0, \tag{3.1}$$

and an element $x \in D_{\mathrm{gl}}(\Gamma)$, the set of lifts

$$x' \in D_{\mathrm{gl}}(\Gamma')$$

extending x is a principal homogeneous space under the action of $D_{\mathrm{gl}}(k[\epsilon])$. Moreover, such a lift exists if and only if a certain obstruction vanishes.

Consider a non-singular complete algebraic curve X over an algebraically closed field k of characteristic $p > 0$. The tangent space of the global deformation functor $D_{\mathrm{gl}}(k[\epsilon])$ is identified with the equivariant cohomology group $H^1(G, X, \mathcal{T}_X)$, which coincides with the invariant space $H^1(X, \mathcal{T}_X)^G$. Furthermore, a local-global theorem exists, characterized by the following short exact sequence:

$$0 \longrightarrow H^1(X/G, \pi_*^G(\mathcal{T}_X)) \longrightarrow H^1(G, X, \mathcal{T}_X) \longrightarrow H^0(X/G, R^1\pi_*^G(\mathcal{T}_X)) \longrightarrow 0 \tag{3.2}$$

$$H^0(X/G, R^1\pi_*^G(\mathcal{T}_X)) \xrightarrow{\cong} \bigoplus_{i=1}^{r} H^1\left(G_{x_i}, \widehat{\mathcal{T}}_{X,x_i}\right)$$

The lifting obstruction is an element of $H^2(G, X, \mathcal{T}_X)$, which decomposes into a direct sum of local cohomology groups $H^2(G_{x_i}, \widehat{\mathcal{T}}_{X,x_i})$ associated with the ramified points $x_i \in X$:

$$H^2(G, \mathcal{T}_C) \cong \bigoplus_{i=0}^{r} H^2(G_{x_i}, \widehat{\mathcal{T}}_{C,x_i}).$$

Here, G_{x_i} represents the isotropy groups and $\widehat{\mathcal{T}}_{X,x_i} = k[[t_i]]\frac{d}{dt_i}$ denotes the completed local tangent spaces. Bertin and Mézard demonstrated that computing these global obstructions can be reduced to the infinitesimal lifting of representations of G_{x_i} into the group of continuous automorphisms $\mathrm{Aut}(k[[t]])$.

3.1. **Comparison with the deformation theory of singular curves: the Bertin-Mézard local-global approach.** The deformation theory of smooth curves with automorphisms exhibits striking formal similarities with the deformation theory of singular curves. These similarities become particularly transparent when viewed through the local-global construction developed by Bertin and Mézard [4], which decomposes the global deformation problem into local deformation problems together with a global compatibility condition.

Local deformation data. Let X be a smooth projective curve over an algebraically closed field k, and let $G \subset \mathrm{Aut}(X)$ be a finite group. In the deformation theory of singular curves, the local deformation theory is controlled by the singular points of the curve: for a singular point $p \in X$, infinitesimal deformations are governed by the local module

$$T^1_p = \mathrm{Ext}^1_{\mathcal{O}_{X,p}}(\Omega_{X,p}, \mathcal{O}_{X,p}),$$

which measures the space of first-order smoothings of the singularity.

In contrast, when X is smooth but equipped with a group action, the role of singular points is played by the *ramification points* of the G-action. For each point $P \in X$ with nontrivial inertia group G_P, Bertin and Mézard associate a local deformation functor D_P, parametrizing G_P-equivariant deformations of the action on the completed local ring $\widehat{\mathcal{O}}_{X,P}$. In positive characteristic, these local deformation problems are often highly nontrivial, reflecting the presence of wild ramification.

Global deformation data. In both theories, the local deformation data does not suffice to determine a global deformation. For singular curves, one must glue local smoothings of singularities to deformations of the normalization. Similarly, in the Bertin-Mézard framework, local deformations of inertia actions must be compatible with a global deformation of the quotient curve

$$Y = X/G.$$

This local-global structure is reflected in an exact sequence of tangent spaces relating the global deformation functor D_{gl} to its local counterparts:

$$0 \longrightarrow H^1(X, T_X)^G \longrightarrow D_{\mathrm{gl}}(k[\varepsilon]) \longrightarrow \bigoplus_{P \in X} D_P(k[\varepsilon]),$$

where the sum runs over the ramification points of the G-action. This sequence is formally analogous to the local-global exact sequences appearing in the deformation theory of singular curves.

Obstructions. The analogy persists at the level of obstruction theory. For singular curves, obstructions arise when local smoothings fail to glue globally and are measured by higher Ext-groups such as $\mathrm{Ext}^2(\Omega_X, \mathcal{O}_X)$. In the equivariant setting, obstructions arise when local lifts of inertia actions cannot be assembled into a global G-action on a deformation of X. These obstructions are typically governed by equivariant cohomology groups, often expressible in terms of group cohomology

$$H^2(G, \mathcal{T}),$$

for suitable sheaves of derivations $\mathcal{T}$.

Conceptual interpretation. From this perspective, a smooth curve with automorphisms behaves like a curve with *stacky* or *orbifold* singularities. Although the curve X itself is nonsingular, the quotient X/G carries orbifold points whose deformation theory closely resembles that of genuine singularities. The Bertin-Mézard local-global construction makes this analogy precise, placing the deformation theory of curves with automorphisms and that of singular curves within a common conceptual framework.

In summary, singularities and automorphisms both localize deformation problems. Singular curves acquire deformation directions from local geometric smoothing, whereas curves with automorphisms are constrained by local equivariant data. In both cases, the global deformation theory is governed by the success or failure of gluing local deformation data into a coherent global object.

4. The tangent space to the global deformation functor

According to [4], the tangent space can be decomposed into local and global contributions, using the low terms sequence corresponding to the composition of global sections and group invariants:

$$0 \to H^1(X/G, \pi_*^G(\mathcal{T}_X)) \to H^1(X, G, \mathcal{T}_X) \to H^0(X/G, R^1\pi_*^G(\mathcal{T}_X)) \to 0$$

- **Global Contribution:** The dimension of the space of deformations that are locally trivial is given by the Riemann-Roch formula on the quotient curve $Y = X/G$:

$$\dim_k H^1(Y, \pi_*^G(\mathcal{T}_X)) = 3g_Y - 3 + \sum_{\mu=1}^{r} \left[\sum_{i=0}^{n_\mu} \frac{e_i^{(\mu)} - 1}{e_0^{(\mu)}} \right],$$

  where g_Y is the genus of Y and $e_i^{(\mu)}$ are the orders of the higher ramification groups at the ramification points b_μ.
- **Local Contribution:** The local contribution equals

$$H^0(C/G, R^1\pi_*^G(\mathcal{T}_C)) \cong \bigoplus_{i=1}^{r} H^1(G_{x_i}, \widehat{\mathcal{T}}_{C,x_i}),$$

  where each summand corresponds to ordinary group cohomology of the groups G_{x_i}. These ramification groups are solvable and one can proceed step by step, using Lyndon-Hochschild-Serre spectral sequence in order to reduce the computation to elementary abelian groups, which can be handled using Artin-Schreier extensions [37]. Unfortunately the kernel of the transgression is difficult to compute so we can only end to an inequality:

$$\dim_k H^1(G_1, \mathcal{T}_0) \leq \sum_{i=1}^{f} \dim_k H^1(G_{t_i}/G_{t_{i-1}}, \mathcal{T}_0^{G_{t_{i-1}}})$$

In [37] the following cases are computed as examples:

- **Fermat Curves:** For the Fermat curve $x_0^{1+p} + x_1^{1+p} + x_2^{1+p} = 0$, the author proves that the tangent space is zero-dimensional ($\dim_k H^1(X, G, \mathcal{T}_X) = 0$), implying the curve is rigid with respect to its maximal automorphism group.

- **Ordinary Curves:** The results recover the previous dimension formulas established by Cornelissen and Kato [12] for ordinary curves.
- **Lehr-Matignon Curves:** Bounds are provided for the dimension of deformations of Lehr-Matignon [47] curves

$$y^p - y = \sum_{i=0}^{m-1} t_i x^{1+p^i} + x^{1+p^m}.$$

4.1. **Tangent space and duality.** In [38] an other approach to the computation of the tangent space to the deformation space is proposed and the the relation between the deformation functor of curves with automorphisms in positive characteristic p and the Galois module structure of holomorphic differentials is investigated. The main results are summarized below:

The Tangent Space and Serre Duality. Let X be a smooth projective curve of genus $g \geq 2$ over an algebraically closed field k of characteristic $p > 0$. Let G be a subgroup of the automorphism group of X. The tangent space $t_{D_{gl}}$ of the global deformation functor for (X, G) is identified with Grothendieck's equivariant cohomology:

$$t_{D_{gl}} \cong H^1(X, G, \mathcal{T}_X),$$

where $\mathcal{T}_X$ is the tangent sheaf of the curve [4].

Proposition 1 (Isomorphism to Invariants). *There is an isomorphism between the equivariant cohomology and the G-invariant subspace of the first Čech cohomology group:*

$$H^1(G, \mathcal{T}_X) \cong H^1(X, \mathcal{T}_X)^G.$$

By applying Serre duality, the above space is related to the space of holomorphic 2-differentials, and the following equality holds:

$$\dim_k H^1(G, \mathcal{T}_X) = \dim_k H^0(X, \Omega_X^{\otimes 2})_G,$$

where $H^0(X, \Omega_X^{\otimes 2})_G$ denotes the module of covariants under the G-action.

Local-Global Principle. Moreover the following homological version of the local-global principle for these deformations is proved:

Proposition 2 (Homological Sequence). *Let $b_1, \ldots, b_r$ be the ramification points of the cover $X \to Y = X/G$ with decomposition groups G_i. The dimension of the space of covariant elements Ω_G is determined by the exact sequence:*

$$0 \to \left(\bigoplus_{i=1}^{r} \mathrm{H}_1(G(b_i), \mathcal{M}^{\otimes 2}/\Omega_{b_i}^{\otimes 2}) \right) \to \Omega_G \to \mathrm{Im}\alpha \to 0,$$

where $\mathrm{Im}\alpha$ *represents the contribution from the quotient curve Y.*

Computations for Cyclic Groups. In the classical case, i.e. in characteristic zero or more general if the action of G on X is tame, [4, p. 206], [37, eq. (40)] the dimension of the tangent space is known to be equal to $3g_Y - 3 + r$, where g_Y denotes the genus of the quotient curve Y and r is the number of branch points. Notice that $3g_Y - 3$ is the dimension of the moduli space of curves of genus g_Y and each branch point adds one further degree of freedom.

In the case of a wild action the situation is more subtle. For the specific case where G is a cyclic group of order p, using results of S. Nakajima the tangent space can be computed:

Theorem 3 (Dimension Formula). *If $G \cong \mathbb{Z}/p\mathbb{Z}$, the dimension is given by:*

$$\dim_k H^0(X, \Omega_X^{\otimes 2})_G = 3g_Y - 3 + \sum_{i=1}^{r} \left\lceil \frac{2(N_i + 1)(p-1)}{p} \right\rceil,$$

where g_Y is the genus of the quotient curve Y, r is the number of branch points, and N_i are constants related to the local ramification filtration. This result recovers the formula previously established by Bertin and Mézard [4].

These results are expanded in [31] as follows. Generalizing a result of Nakajima [51, Theorem 1] from the case when G is cyclic of order p to the case when G is an arbitrary cyclic p-group, Borne [5, Theorem 7.23] has explicitly determined the Galois module structure of $H^0(X, \mathcal{O}_X(D))$ for any G-invariant divisor D on X of degree greater than $2g_X - 2$. Although the formulation of Borne's theorem requires quite involved definitions and is in particular difficult to state, its consequence for the dimension of $H^0(X, \Omega_X^{\otimes 2})_G$ is simple:

$$\dim_k H^0(X, \Omega_X^{\otimes 2})_G = 3g_Y - 3 + \sum_{j=1}^{r} \left\lfloor \frac{2d(P_j)}{e_0(P_j)} \right\rfloor .$$

This result can also be derived from core results in [4] and becomes [4, Proposition 4.1.1] if G is cyclic.

If we assume that the action of G on X is weakly ramified, *i.e.*, that the second ramification group $G_2(P)$ vanishes for all $P \in X$, we can prove

$$\dim_k H^0(X, \Omega_X^{\otimes 2})_G = 3g_Y - 3 + \sum_{j=1}^{r} \log_p |G(P_j)| + \begin{cases} 2r & \text{if } p > 3 \\ r & \text{if } p = 2 \text{ or } 3, \end{cases}$$

see [31, Th. 3.1]. The above formula matches up with results of Cornelissen and Kato [28, Thm 4.5, Thm 5.1(b)] for the deformation of ordinary curves. The results of article [31] is based on results of B. Köck [30, Thm 4.5] on week ramified covers and their relation with projective modules.

5. The lifting problem

Let k be a field of prime characteristic $p > 0$. *A lift of k to characteristic 0* is the field of fractions L of any integral extension of the ring of Witt vectors $W(k)$, a classical construction by Witt [76] that generalizes the p-adic integers $\mathbb{Z}_p = W(\mathbb{F}_p)$. We will consider the case of algebraically closed fields k. Note that integral extensions of $W(k)$ are discrete valuation rings of mixed characteristic, with residue field k.

Consider a projective, non-singular curve $\mathcal{X}_0$ over k and let R be an integral extension of $W(k)$. *A lift of* $\mathcal{X}_0/k$ *to characteristic* 0, is a curve $\mathcal{X}_\eta$ over $L = \mathrm{Quot}R$, obtained as the generic fibre of a flat family of curves $\mathcal{X}/R$ whose special fibre is $\mathcal{X}_0/k$. Such lifts have been extensively used by arithmetic geometers to reduce characteristic p problems to the, much better understood, characteristic 0 case.

One of the earliest uses of the idea of lifting is the approach of J.P. Serre [69] in an attempt to define an appropriate cohomology theory, which could solve the Weil conjectures. The lifting of an algebraic variety to characteristic zero is unfortunately not always possible and Serre was able to give such an example, see [70]. The progress made in deformation theory by Schlessinger [65] identified the lifting obstruction as an element in $H^2(X, T_X)$, see [68, 1.2.12], [24, 5.7 p.41].

An other motivation for the lifting problem was the generalization of Riemann's existence theorem for Riemann surfaces to algebraic curves of positive characteristic. Riemann's existence theorem asserts that every finite topological branched cover $f^{top} : Y^{top} \to X(\mathbb{C})$ of a smooth projective complex curve X originates from a unique algebraic branched cover $f : Y \to X$. Specifically, the corresponding topological cover f^{an} of the algebraic cover is homeomorphically equivalent to the original topological cover f^{top} through a map i_Y that respects the covering structure $f^{top} \circ i_Y = f^{an}$.

In positive characteristic the situation is different. The Artin-Schreier covers are étale covers of $\mathbb{P}^1 - \{\infty\}$, a non-possible situation in characteristic zero, since $\mathbb{P}^1(\mathbb{C}) - \{\infty\} \cong \mathbb{C}$, is simply connected.

Consider a complete discrete valuation ring R with residue field k (char $p > 0$, algebraically closed) and fraction field K. Let X_R be a smooth projective relative curve over R equipped with n disjoint marking sections $\{x_{i,R}\}$. Grothendieck's theory of tame lifting ensures that any finite tamely ramified cover $f : Y \to X$ of the special fiber, étale outside the markings $\{x_i\}$, admits a unique lift to a finite flat branched cover $f_R : Y_R \to X_R$ over R. This lifting f_R is uniquely determined by the requirement that it remains étale outside the sections $\{x_{i,R}\}$ and preserves the ramification indices of f. Furthermore, the lifting functor preserves the G-Galois property.

If f is étale, the theorem follows from Grothendieck's theory of étale lifting ([21, I, Corollaire 8.4], combined with [21, III, Prop. 7.2]). In general, we can use Grothendieck's theory of tame lifting ([21, XIII, Corollaire 2.12], see also [75].

5.1. **Lifts of curves with automorphisms.** Let $\mathcal{X}_0/k$ be a projective, non-singular curve as in the previous section. Such a curve can always be lifted to characteristic zero, since the obstruction lives in the second cohomology which is always zero for curves. However, one might ask if it is possible to deform the curve together with its automorphism group, see [4]. This is not always possible, since Hurwitz's bound for the order of automorphism groups in characteristic 0 ensures that the answer for a general group G is negative, see [17][53]. In the same spirit, J. Bertin in [3] provided an obstruction for the lifting based on the Artin representation which vanishes for cyclic groups. Note that, even in positive characteristic, the order of cyclic automorphism groups is bounded by the classical Hurwitz bound, see [52]. The existence of such a lift for cyclic p-groups was conjectured by Oort in [59] and was laid to rest three decades later by Obus-Wewers [58] and Pop [62].

In the meantime, the case for $G = \mathbb{Z}/p\mathbb{Z}$ was studied by Oort himself and Sekiguchi-Suwa [66, 67], who unified the theory of cyclic extensions of the projective line in characteristic p (*Artin-Schreier extensions*) and that of cyclic extensions of the projective line in characteristic 0 (*Kummer extensions*). The unified theory is usually referred to as *Kummer-Artin Schreier-Witt* theory or *Oort-Sekiguchi-Suwa* (OSS) theory. Using these results, Bertin-Mézard in [4] provided an explicit description of the affine model for the Kummer curve in terms of the affine model for the Artin-Schreier curve. Following this construction, members of our research group in [26] proposed the study of the Galois module structure of the relative curve $\mathcal{X}/R$. As a byproduct, they found an explicit basis of the R-module of relative holomorphic differentials $H^0(\mathcal{X}, \Omega_{\mathcal{X}})$, using Boseck's work [6] on holomorphic differentials.

5.2. **Explicit liftings of curves.**

5.2.1. *The Bertin-Mezard family.* Let k be an algebraically closed field of characteristic $\mathrm{char}(k) = p > 0$. Denote by $W(k)[\zeta]$ the ring of Witt vectors over k extended by a p-th root of unity ζ and let $\lambda = \zeta - 1$. By [25, sec. 8.10] $W(k)[\zeta]$ is a discrete valuation ring with maximal ideal $\mathfrak{m}$ and residue field isomorphic to k. Let $m \geq 1$ be a natural number not divisible by p; for any $1 \leq \ell \leq p-1$ we write $m = pq - \ell$ and consider, as in [26, sec. 3], the local ring

$$R = \begin{cases} W(k)[\zeta][[x_1, \ldots, x_q]] & \text{if } \ell = 1 \\ W(k)[\zeta][[x_1, \ldots, x_{q-1}]] & \text{if } \ell \neq 1 \end{cases}$$

with maximal ideal $\mathfrak{m}_R = \langle \mathfrak{m}, \{x_i\} \rangle$. We write

$$K = \mathrm{Quot}\,(R/\mathfrak{m}) = \begin{cases} \mathrm{Quot}\,(k[[x_1, \ldots, x_q]]) & \text{if } \ell = 1 \\ \mathrm{Quot}\,(k[[x_1, \ldots, x_{q-1}]]) & \text{if } \ell \neq 1 \end{cases}$$

and consider the extension of the rational function field $K(x)$ given by $\mathcal{X}_0 : X^p - X = \frac{x^\ell}{a(x)^p}$, where

$$a(x) = \begin{cases} x^q + x_1 x^{q-1} + \cdots + +x_{q-1}x + x_q & \text{if } \ell = 1 \\ x^q + x_1 x^{q-1} + \cdots + x_{q-1}x & \text{if } \ell \neq 1. \end{cases} \tag{5.1}$$

Bertin-Mézard proved in [4, sec. 4.3] that the curve above lifts to a curve over $L = \mathrm{Quot}(R)$ given by

$$\mathcal{X}_\eta : y^p = \lambda^p x^\ell + a(x)^p$$

for $y = a(x)(\lambda X + 1)$, which is the normalization of $R[x]$ in $L(y)$. This gives rise to a family $\mathcal{X} \to \mathrm{Spec}(R)$, with special fibre $\mathcal{X}_0$ and generic fibre $\mathcal{X}_\eta$:

$$\begin{array}{ccccc} \mathrm{Spec}(k) \times_{\mathrm{Spec}(R)} \mathcal{X} = \mathcal{X}_0 & \hookrightarrow & \mathcal{X} & \hookleftarrow & \mathcal{X}_\eta = \mathrm{Spec}(L) \times_{\mathrm{Spec}(R)} \mathcal{X} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \hookrightarrow & \mathrm{Spec}(R) & \hookleftarrow & \mathrm{Spec}(L) \end{array} \tag{5.2}$$

In [26] the study of spaces of holomorphic polydifferentials of a relative curve $\mathcal{X} \to \mathrm{Spec}(R)$ as a $R[G]$-module is proposed. Moreover the $R[G]$-module structure of holomorphic differentials is computed. We now describe the result of this article.

Proposition 4. *Let S be an integral domain that is a $W(k)[\zeta]$-algebra. Set $\lambda = \zeta - 1$. For $a_0, a_1 \in \mathbb{Z}$, $1 \leq a_1 \leq p$, we consider the S-module*

$$V'_{a_0,a_1} := S\langle (\lambda X + 1)^i : a_0 \leq i < a_0 + a_1 \rangle \subset S(\lambda X + 1).$$

Consider the diagonal integral representation of a cyclic group of order p on V'_{a_0,a_1} by defining

$$\sigma(\lambda X+1)^i=\zeta^i(\lambda X+1)^i. \tag{1}$$

Notice that the modules V'_{a_0,a_1} and V'_{a_0+p,a_1} are isomorphic as G-modules. The problem with the module V'_{a_0,a_1} is that there is no good reduction of it modulo the maximal ideal of S. So we define the S-module

$$V_{a_0,a_1}:=S\langle(\lambda X+1)^{a_0}X^i:0\leq i<a_1\rangle\subset S(\lambda X+1,X)$$

instead. The two modules are $\mathrm{GL}_{a_1}(\mathrm{Quot}(S))$-equivalent but not $\mathrm{GL}_{a_1}(S)$-equivalent. After a $\mathrm{GL}_{a_1}(\mathrm{Quot}(S))$ change of basis the representation on V_{a_0,a_1} becomes equivalent to

$$\rho(\sigma)=\mathrm{diag}(\zeta^{a_0},\zeta^{a_0+1},\ldots,\zeta^{a_0+a_1-1})A_{a_1}. \tag{2}$$

The $S[G]$-representation V_{a_0,a_1} is indecomposable.

Definition 1. *Define by V_a the indecomposable integral representation $V_{1-p,a}$ of Proposition 4. It has the matrix representation given in (2). The module V_a is free of rank a.*

Theorem 5. *Let σ be an automorphism of $\mathcal{X}$ of order $p\neq 2$ and conductor m with $m=pq-l$, $1\leq q$, $1\leq l\leq p-1$. Let*

$$R=\begin{cases}W(k)[\zeta][[x_1,\ldots,x_q]] & \textit{if } l=1,\\ W(k)[\zeta][[x_1,\ldots,x_{q-1}]] & \textit{if } l\neq 1,\end{cases} \tag{3}$$

be the Oort-Sekiguchi-Suwa factor (see also Theorem 9) of the versal deformation ring R_σ. The free R-module $H^0(\mathcal{X},\Omega_{\mathcal{X}})$ of relative differentials has the following $R[G]$-structure:

$$H^0(\mathcal{X},\Omega_{\mathcal{X}})=\bigoplus_{\nu=0}^{p-2}V_\nu^{\delta_\nu}, \tag{4}$$

where

$$\delta_\nu=\begin{cases}q+\left\lceil\frac{(\nu+1)l}{p}\right\rceil-\left\lceil\frac{(2+\nu)l}{p}\right\rceil & \textit{if } \nu\leq p-3,\\ q-1 & \textit{if } \nu=p-2.\end{cases} \tag{5.3}$$

5.3. **The Oort conjecture.** The *lifting problem* studies the following construction: given a smooth, projective curve X in characteristic p acted on by a finite group G, does there exist an arithmetic surface $\mathcal{X}$ over a ring R of mixed characteristic whose special fiber is G-isomorphic to X? The answer in general is no! The genus has to remain constant in fibres and there are examples of curves (like the Hermitian curve $x^{1+p}+y^{1+p}+z^{1+p}=0$) with order of the automorphism group $>84(g-1)$ while in characteristic zero the order of the automorphism group is bounded from above by $84(g-1)$ (Hurwitz bound). These obstructions do not appear if G is abelian since Nakajima proved [52] that a Hurwitz like bound hold always for abelian groups. Oort conjecture states that such a lifting from characteristic p to characteristic zero is always possible when the group G is cyclic.

The question has its origins in Serre's Mexico paper [69], where the author introduced lifting techniques in an early attempt at the Weil conjectures. By a result of Grothendieck [21, XIII,2.12], if $\mathrm{char}(k)\nmid |G|$, then X with the action of G can always be lifted to characteristic 0; this implies in particular that lifting a curve without a group action is always possible. The case $\mathrm{char}(k)\mid |G|$ proved more challenging and remains open. In [59], F. Oort conjectured that if G is cyclic then the pair (X,G) can always be lifted. Following a series of developments, [66],[4], [17], Obus-Wewers in [58] and Pop in [62] proved Oort's conjecture and thus paved the road to the following question: for which other groups G can the pair (X,G) be lifted to characteristic 0? A series of lifting obstructions, such as [3], [11] and [8], reduced the candidates to the following shortlist: G must be either cyclic, dihedral of order $2p^n$ or the alternating group A_4. The cyclic case was settled in [58] and [62], while A_4 is due to Obus [56], leaving the case for dihedral groups; D_p is due to Bouw-Wewers [7] and Pagot [60], D_4 is due to Weaver [74], D_9 is due to Obus [57], while D_{25} and D_{27} are due to joint work of Dr. Karagiannis, member of our team [13]. These are the only dihedral groups known to satisfy the lifting problem. The PI with A. Terezakis, in a series of articles [41], [41], [39] provided a counterexample to the generalized Oort conjecture. The problem now is: which are exactly the groups D_{p^h} that can be lifted to characteristic zero?

5.4. **Lifting of representations.** Let $\mathcal{G} : \mathcal{C} \to \text{Groups}$ be a group functor, see [14, ch. 2]. We will be mainly interested in two group functors. The first one, GL_g, will be represented by the by the group scheme $G_g = \Lambda[x_{11}, \ldots, x_{gg}, \det(x_{ij})^{-1}]$, that is $\mathrm{GL}_g(\Gamma) = \mathrm{Hom}_\Lambda(G_g, \Gamma)$. The second one is the group functor from the category of rings to the category of groups $\mathcal{N} : \Gamma \mapsto \mathrm{Aut}\Gamma[[t]]$.

We also assume that each group $\mathcal{G}(\Gamma)$ is embedded in the group of units of some ring $\mathcal{R}(\Gamma)$ depending functorially on Γ. This condition is imposed since our argument requires us to be able to add certain group elements. We also assume that the additive group of the ring $\mathcal{R}(\Gamma)$ has the structure of direct product Γ^I, while $\mathcal{R}(\Gamma) = \mathcal{R}(\Lambda) \otimes_\Lambda \Gamma$. Notice, that I might be an infinite set, but since all rings involved are Noetherian Γ^I is flat, see [46, 4F]. We will employ this machinery in the following section.

6. Harbater-Katz-Gaber covers

Let k be an algebraically closed field of characteristic $p > 0$, and let G be a finite group. A *generalized Harbater-Katz-Gabber G-curve* is a smooth, projective, connected curve C over k endowed with a faithful action of G such that the quotient C/G is isomorphic to $\mathbb{P}^1_k$ and the natural morphism

$$\pi : C \longrightarrow \mathbb{P}^1_k$$

is branched at at most two points. One of these branch points corresponds to a totally wildly ramified point, while the other branch point (if present) is tamely ramified. Equivalently, the inertia group at the wild branch point has a p-subgroup as G_1, whereas the inertia group at the tame branch point is cyclic of order prime to p. In the case where only one branch point occurs, the cover is necessarily totally wildly ramified at a unique branched point and the inertia subgroup is a p-group.

The Harbater-Katz-Gabber compactification theorem [22], [29], asserts that there is a HKG-cover $X_{HKG} \to \mathbb{P}^1$ ramified only at one point P of X with Galois group $G = \mathrm{Gal}(X_{HKG}/\mathbb{P}^1) = G_0$ such that $G_0(P) = G_0$ and the action of G_0 on the completed local ring $\hat{\mathcal{O}}_{X_{HKG},P}$ coincides with the original action of G_0 on $\mathcal{O}$.

By considering the Harbater-Katz-Gabber compactification to an action on the local ring $k[[t]]$, we have the advantage to attach global invariants, like genus, p-rank, differentials etc., in the local case. Also finite subgroups of the automorphism group $\mathrm{Aut}k[[t]]$, which is a difficult object to understand (and is a crucial object in understanding the deformation theory of curves with automorphisms, see [4]) become subgroups of $\mathrm{GL}(V)$ for a finite dimensional vector space V.

The HKG curves provide global realizations of prescribed local G-extensions with both wild and tame ramification and play an important role in the study of local-global principles, deformation theory of Galois actions, and lifting problems for curves and covers in characteristic p.

Notice that in the literature this compactification is called by the name "Katz-Gabber" but M. Matignon pointed to us that it was D. Harbater who first considered this construction in his work [22] for the case of p-groups; his results were later generalized by N. Katz in [29]. From now on we will call these Harbater-Katz-Gabber covers as HKG-covers.

6.1. **The passage from $\mathrm{Aut}k[[t]]$ to $\mathrm{GL}_n(k)$.** The representation

$$\rho : G(P) \longrightarrow \mathrm{Aut}k[[t]]$$

plays an essential role in ramification filtration and in deformation of curves with automorphisms. The group $\mathrm{Aut}k[[t]]$ is a difficult group to understand compared to the group $\mathrm{Aut}(V) = \mathrm{GL}_{\dim(V)}(k)$.

The article [35] was the first attempt of members of our research group to replace automorphisms of formal powerseries with automorphisms of vector spaces.

One of the core results of this article is the following: Let $1 \leq m \leq 2g-1$ be the smallest pole number not divisible by the characteristic. There is a faithful representation

$$\rho : G_1(P) \longrightarrow \mathrm{GL}(mP).$$

This result makes $G_1(P)$ realizable as a finite algebraic subgroup of the linear group $\mathrm{GL}_{\ell(mP)}(k)$. Moreover the flag of vector spaces $L(iP)$ for $i \leq m$ is preserved, so the representation matrices are upper triangular, or in other words $G_1(P)$ is a subgroup of the Borel group of the flag.

We assume that $m = m_0 > m_1 > \cdots > m_r = 0$, are the pole numbers $\leq m$. Therefore, a basis for the vector space $L(mP)$ is given by

$$\left\{ 1, \frac{u_i}{t^{m_i}}, \frac{1}{t^m} : \text{where } 1 < i < r, p \mid m_i \text{ and } u_i \text{ are certain units} \right\}.$$

With respect to this basis, an element $\sigma \in G_1(P)$ acts on $1/t^m$ by

$$\sigma \frac{1}{t^m} = \frac{1}{t^m} + \sum_{i=1}^{r} c_i(\sigma) \frac{u_i}{t^{m_i}},$$

and then it maps the local uniformizer t to

$$\sigma(t) = \frac{\zeta t}{(1 + \sum_{i=1}^{r} c_i(\sigma) u_i t^{m-m_i})^{1/m}}, \tag{4}$$

where ζ is an m-th root of 1.

The above expression can be written in terms of a formal power series as:

$$\sigma(t) = \zeta t \left(1 + \sum_{\nu \geq 1} a_\nu(\sigma) t^\nu \right). \tag{5}$$

Since σ is an automorphism of the power series ring and $\sigma^{|G_1(P)|} = 1$, and $(m,p) = 1$ we obtain that $\zeta = 1$ and (5) can be written as:

$$\sigma(t) = t \left(1 + \sum_{\nu \geq 1} a_\nu(\sigma) t^\nu \right). \tag{6}$$

The above computation allows us to compute the jumps in the filtration of the group $G_1(P)$.

Proposition 6. *If $g \geq 2$ and $p \neq 2,3$, then there is at least one pole number $m_r \leq 2g-1$ not divisible by the characteristic p. Then there is a faithful representation*

$$\rho : G_1(P) \to \mathrm{GL}\big(L(m_r P)\big).$$

Proposition 7. *Let X be a curve acted on by the group G. For every fixed point P on X we consider the corresponding faithful representation defined in proposition 6:*

$$\rho : G_1(P) \to \mathrm{GL}_{\ell(m_r P)}(k).$$

If $G_i(P) > G_{i+1}(P)$, for $i \geq 1$, then $i = m_r - m_k$, for some pole number m_k.

6.2. **Representation filtration and HKG-covers.** In positive characteristic the group $G(P)$ admits the following ramification filtration:

$$G(P) \supseteq G_0(P) \supseteq G_1(P) \supseteq G_2(P) \supseteq \cdots \supseteq \{\mathrm{id}\},$$

where

$$G_i(P) = \{\sigma \in G(P) : v_P(\sigma(t) - t) \geq i+1\},$$

for a local uniformizer t at P and v_P its corresponding valuation. Notice that $G_1(P)$ is the p-part of $G(P)$.

In [27] the notion of representation filtration is introduced in order to determine the jumps of the ramification filtration, i.e. to find the indices such that $G_i(P) \gneq G_{i+1}(P)$. This is a deep question related to the structure of $G_1(P)$ and to the curve in question. For instance if $G_1(P)$ is abelian then the Hasse-Arf theorem [71, Theorem p. 76] imposes very strong divisibility relations among the jumps. Let us fix the notation for the jumps of the ramification filtration, in what follows $G_0(P)$ is a p-group so:

$$G_0(P) = G_1(P) = G_{b_1} > G_{b_2} > \cdots > G_{b_\mu} > \{\mathrm{id}\}.$$

This means that $G_{b_\nu} \supsetneq G_{b_\nu+1}$ for every $1 \leq \nu \leq \mu$ and that there are μ jumps.

6.3. **Weierstrass semigroups.** On the other hand, the Weierstrass semigroup at P consists of all elements of the function field of the curve that have a unique pole at P. More precisely we can consider the flag of vector spaces

$$k = L(0) = L(P) = \cdots = L((i-1)P) < L(iP) \leq \cdots \leq L((2g-1)P),$$

where

$$L(iP) := \{f \in F : \mathrm{div}(f) + iP \geq 0\} \cup \{0\}.$$

We will write $\ell(D) = \dim_k L(D)$, for a divisor D. An integer i will be called a pole number if there is a function $f \in F^*$ so that $(f)_\infty = iP$, or equivalently $\ell\big((i-1)P\big) + 1 = \ell\big(iP\big)$. If i is not a pole number, we call it a gap. The set of pole numbers at P form a numerical semigroup $H(P)$ which is called the Weierstrass semigroup at P. Note that 0 is always a pole number; thus, from now on when we write $H(P)$ we always assume that $\{0\} \in H(P)$ for every Weierstrass semigroup. It is known that there are exactly g pole numbers that are smaller or equal to $2g-1$ and that every integer $i \geq 2g$ is in the Weierstrass semigroup, see [73, I.6.7]. The case of symmetric Weierstrass semigroups is an extreme example where the maximum gap equals $2g-1$.

It is also known that there is a close connection between the group $G(P)$ and the Weierstrass semigroup at P. I. Morisson and H. Pinkham [50] studied this connection in characteristic zero for *Galois Weierstrass points*: a point P on a compact Riemann surface Y is called Galois Weierstrass if for a meromorphic function f on Y such that $(f)_\infty = dP$, where d is the least pole number in the Weierstrass semigroup at P, the function $f : Y \to \mathbb{P}^1(\mathbb{C})$ gives rise to a Galois cover. We generalise [50] in positive characteristic. Notice that the first non zero element in $H(P)$ is not enough to grasp the group structure. We have to go up to the first pole number in $H(P)$ that is not divisible by p to do so. And of course the stabilizer $G(P)$ and its p-part $G_1(P)$ do not have to be cyclic groups anymore.

6.3.1. *Action on Riemann-Roch spaces.*

Definition 2. *Let m_r be the smallest pole number at P not divisible by the characteristic. Denote with*

$$0 = m_0 < \cdots < m_{r-1} < m_r$$

all the pole numbers at P in increasing sequence which are $\leq m_r$. From now on, $f_i \in F$, with $0 \leq i \leq r$ will denote the functions such that $(f_i)_\infty = m_i P$.

As we have already proved in [35, Lemmata 2.1, 2.2] there is a faithful action of the p-part $G_1(P)$ of the decomposition group $G(P)$ on the spaces $L(iP)$ 6, 7.

In this section we consider HKG p-covers: G is a p-subgroup of $\mathrm{Aut}(X_{HKG})$, i.e. $G_0 = G_1$, while P will always be the unique ramified point of the cover, i.e. $G = G(P)$, which moreover ramifies totally, i.e. $G_0 = G$.

From now on we will focus on HKG p-covers and we will use the notation X for such a curve, instead of X_{HKG}, unless otherwise stated.

6.3.2. *Representation Filtration.* Recall that

$$0 = m_0 < \cdots < m_{r-1} < m_r$$

are all the pole numbers at P in increasing sequence up to m_r.

Definition 3. *For each $0 \leq i \leq r$ we consider the representations*

$$\rho_i : G_1(P) \to \mathrm{GL}(L(m_i P)).$$

We form the decreasing sequence of groups:

$$G_1(P) = \ker\rho_0 \supseteq \ker\rho_1 \supseteq \ker\rho_2 \supseteq \cdots \supseteq \ker\rho_r = \{1\}. \tag{6.1}$$

We will call this sequence of groups "the representation" filtration.

Note that the spaces $L(m_i P)$ are fixed by the action of $G_1(P)$. The filtration of eq. (6.1) leads to a successive sequence of elementary abelian p-group extensions of the field $F^{G_1(P)}$:

$$F^{G_1(P)} = F^{\ker\rho_0} \subseteq F^{\ker\rho_1} \subseteq F^{\ker\rho_2} \subseteq \cdots \subseteq F^{\ker\rho_r} = F. \tag{6.2}$$

We call an index i a *jump of the representation filtration* if and only if $\ker \rho_i \supsetneq \ker \rho_{i+1}$. Let us also fix the notation for the representation jumps:

$$G_1(P) = \ker \rho_0 = \cdots = \ker \rho_{c_1} > \cdots > \ker \rho_{c_{n-1}} > \ker \rho_{c_n} > \{\mathrm{id}\}.$$

The above sequence of groups jumps at say n certain integers, we call them the jumps of the representation filtration,

$$c_1 < c_2 < \cdots < c_{n-1} < c_n = r-1,$$

Remark 8. *Every element $\sigma \in \ker\rho_i$ fixes by definition all f_ν such that $(f_\nu)_\infty = m_\nu P$ for $\nu \leq i$. A non negative integer i is a jump whenever the function f_{i+1} is not $\ker \rho_i$ invariant.*

If c_i is a representation jump then m_{c_i+1} is a minimal generator of $H(P)$. Since the sequence of the groups $\ker \rho_{c_i}$ jumps, the corresponding sequence of fields will also jump and, moreover,

$$F^{\ker \rho_{c_{i+1}}} = F^{\ker \rho_{c_i}}(f_{c_i+1}). \tag{6.3}$$

Definition 4. *In order to simplify notation we set $F_i = F^{\ker \rho_{c_i}}$, $\bar{m}_i = m_{c_i+1}$ and $\bar{f}_i = f_{c_i+1}$. Denote also by $p^{h_i} = |\ker \rho_{c_{i+1}}|$, for all $1 \leq i \leq n-1$, and $p^{h_0} = G_1(P)$.*

So eq. (6.3) can be written as $F_{i+1} = F_i(\bar{f}_i)$. Thus in every extension we add an extra function $f_{c_i+1} = \bar{f}_i$ which in turn adds a new generator $\bar{m}_i$ in the previous semigroup. Define $Q_i = F_i \cap P$ for $1 \leq i \leq n+1$ to be the *unique* ramification points of the tower defined in eq. (6.2).

We have the following picture of fields, groups, places and semigroups

$$\begin{array}{cccc}
F & \{1\} & P & H(P) \\
| & | & | & | \\
F_{i+1} = F^{\ker \rho_{c_{i+1}}} & \ker \rho_{c_{i+1}} & Q_{i+1} & H(Q_{i+1}) = \langle p^{h_{i-1}-h_i} H(Q_i), \lambda_i \rangle_{\mathbb{Z}_+} \\
| & | & | & | \\
F_i = F^{\ker \rho_{c_i}} & \ker \rho_{c_i} & Q_i & H(Q_i) \\
| & | & | & | \\
F_0 = F^{G_1(P)} & G_1(P) & Q_1 & \mathbb{Z}_+
\end{array}$$

Using the relation of the semigroups in Galois extension of function fields we will see that the semigroup of F_2 at Q_2 is

$$\Sigma_2 := \left| \frac{\ker \rho_{c_1}}{\ker \rho_{c_2}} \right| \mathbb{Z}_+ + \lambda_1 \mathbb{Z}_+ = p^{h_0 - h_1} \mathbb{Z}_+ + \lambda_1 \mathbb{Z}_+$$

with $(\lambda_1, p) = 1$.

Notice that, $\lambda_1 = 1$ if and only if $F^{\ker \rho_{c_2}}$ is rational. We proceed in this way and we have that

$$\Sigma_{i+1} = p^{h_{i-1}-h_i} \Sigma_i + \lambda_i \mathbb{Z}_+, \text{ for all } 1 \leq i \leq n,$$

i.e. the semigroup of a field at Q_{i+1} in the sequence given in eq. (6.2) is the semigroup of the previous field at Q_i multiplied by the order of their Galois group, plus an extra element λ_i prime to p and all their $\mathbb{Z}_+$ linear combinations. The elements

$$p^{h_1}\lambda_1 < p^{h_2}\lambda_2 < \cdots < p^{h_{n-1}}\lambda_{n-1} < \lambda_n = \frac{m_{c_n+1}}{|\ker \rho_{c_{n+1}}|} = m_r,$$

are inside the set of generators of the Weierstrass semigroup at P and that if we add the element p^{h_0} then,

$$\langle p^{h_0}, p^{h_1}\lambda_1, \ldots, p^{h_{n-1}}\lambda_{n-1}, \lambda_n \rangle_{\mathbb{Z}_+} = H(P).$$

Now we are ready to state our two main theorems: Concerning the structure of $H(P)$, the Weierstrass semigroup at P, we have the following

Theorem 9. *(1) For every jump of the representation filtration c_i, $1 \leq i \leq n$ there exists a generator of $H(P)$ of the form $\bar{m}_i = m_{c_i+1} = p^{h_i}\lambda_i$, where $(\lambda_i, p) = 1$.*
(2) The first ramification jump affects the structure of $H(P)$ in the following way:
(a) If $G_1(P) > G_2(P)$, then the extension $F/F^{G_2(P)}$ is also HKG, and the Weierstrass semigroup $H(P)$ is minimally generated by $\bar{m}_i$, with $1 \leq i \leq n$. Moreover $|G_2(P)| = \bar{m}_1 = m_1$.
(b) If $G_1(P) = G_2(P)$ then we need $\bar{m}_i$, $1 \leq i \leq n$ together with $p^{h_0} = |G_1(P)|$ in order to generate $H(P)$. In this case $|G_1(P)| \neq \bar{m}_i$ for all $1 \leq i \leq n$.
In both cases the semigroup $H(P)$ is symmetric.

The relation of the representation with the ramification filtration is given in terms of the following:

Theorem 10. *Assume that $X \to X/G_1(P) = \mathbb{P}^1$ is a HKG-cover. Then*
(1) The jumps of the ramification filtration are the integers λ_i for $1 \leq i \leq n$, i.e. $\lambda_i = b_i$ for every such i, while the number of ramification and representation jumps coincide, i.e. $\mu = n$.
(2) $G_{b_i} = \ker \rho_{c_i}$ for all $2 \leq i \leq n$.

6.4. **Cohomological Approach to HKG-covers.** In [43] a *cohomological classification* of HKG-covers are studied in terms of explicit H^1-classes satisfying compatibility conditions, together with applications to the structure of finite subgroups of the Nottingham group.

Let $P \in X$ be the unique ramified point. The lower ramification filtration

$$G_1(P) \supseteq G_2(P) \supseteq \cdots$$

is related to the *representation filtration* arising from the natural action of G on the Riemann-Roch spaces

$$L(mP), \qquad m \in H(P),$$

where $H(P)$ is the Weierstrass semigroup at P.

A key result is that, for HKG-covers, the jumps of the ramification filtration coincide with the jumps of the representation filtration. This allows one to write the function field $k(X)$ as a tower

$$F_0 \subset F_1 \subset \cdots \subset F_{s+1} = k(X),$$

where each extension F_{i+1}/F_i is *elementary abelian*.

Let $\bar{m}_1, \ldots, \bar{m}_s$ be the generators of the Weierstrass semigroup and $\bar{f}_i$ functions with pole divisor $(\bar{f}_i)_\infty = \bar{m}_i P$. It is proved that for any $m \in H(P)$,

$$L((m-1)P) = k_{n,m}[\bar{f}_0, \bar{f}_1, \ldots, \bar{f}_s],$$

where the right-hand side consists of bounded-degree monomials in the $\bar{f}_i$, with explicit bounds determined by the ramification filtration. This description is fundamental for making the group action explicit.

For each generator $\bar{f}_i$, the action of G is given by

$$\sigma(\bar{f}_i) = \bar{f}_i + \bar{C}_i(\sigma),$$

where

$$\bar{C}_i \in Z^1\big(G,\, k_{n,\bar{m}_i}[\bar{f}_0, \ldots, \bar{f}_{i-1}]\big).$$

The central result of the paper is:

Theorem 11. *An HKG-cover is completely determined by a sequence of cohomology classes*

$$\bar{C}_i \in H^1\big(G,\, k_{n,\bar{m}_i}[\bar{f}_0, \ldots, \bar{f}_{i-1}]\big)$$

satisfying the compatibility conditions

$$P_i(\bar{C}_i) = 0,$$

where P_i is an additive polynomial describing the elementary abelian extension F_{i+1}/F_i. Conversely, every such compatible sequence defines a unique HKG-cover.

The compatibility condition expresses the fact that F_{i+1}/F_i is a generalized Artin-Schreier extension

$$P_i(\bar{f}_i) = D_i, \qquad D_i \in F_i.$$

This provides a non-abelian analogue of the classical Artin-Schreier-Witt theory.

Since the choice of generators $\bar{f}_i$ is not canonical, it is proved that the intrinsic data of an HKG-cover is given by projective cohomology classes

$$[\bar{C}_i] \in \mathbb{P}H^1\big(G, L(\bar{m}_i P)\big),$$

satisfying induced compatibility conditions. This yields a coordinate-free classification of HKG-covers.

6.4.1. *Application to the Nottingham Group.* Using the last generator $\bar{f}_s$, whose pole order is the first integer prime to p, canonical local parameter $t = \bar{f}_s^{-1/m}$ is defined. Moreover every finite p-subgroup G of the Nottingham group acts (after conjugation) by

$$\sigma(t) = t\big(1 + \bar{C}_s(\sigma)t^m\big)^{-1/m}, \qquad \sigma \in G.$$

Thus, the conjugacy class of $G \subset \mathrm{Aut}(k[[t]])$ is encoded by the final cohomology class $\bar{C}_s$. This yields an explicit description of finite p-subgroups and recovers classical normal forms for elements of order p^h.

Thus:

- HKG-covers are governed by cohomological data rather than equations;
- Non-abelian p-group actions admit a stepwise Artin-Schreier description;
- Finite subgroups of the Nottingham group can be classified via H^1-data.

This places HKG-covers firmly within the cohomological philosophy of I. Shafarevich [72] and provides a powerful bridge between algebraic curves, ramification theory, and local group actions.

7. The canonical ideal, Petri's theorem, syzygies

7.1. **Syzygies acted by automorphisms.** The theory of syzygies which originates in the work of Hilbert and Sylvester has attracted a lot of researchers and it seems that a lot of geometric information can be found in the minimal free resolution of the ring of functions of an algebraic curve. For an introduction to this fascinating area we refer to [16].

In [42] it is proved that the automorphism group G of the curve acts linearly on a minimal free resolution $\mathbf{F}$ of the ring of regular functions S_X of the curve X canonically embedded in $\mathbb{P}^{g-1}$. Notice that an action of a group G on a graded module M gives rise to a series of linear representations $\rho_d : G \to M_d$ to all linear spaces M_d of degree d, for $d \in \mathbb{Z}$. For the case of the free modules F_i of the minimal free resolution $\mathbf{F}$ we relate the actions of the group G in both F_i and in the dual F_{g-2-i} in terms of an inner automorphism of G.

This information is used in order to show that the action of the group G on generators of the modules F_i sends generators of degree d to linear combinations of generators of degree d. Let $S = \mathrm{Sym}(H^0(X, \Omega_X))$ be the symmetric algebra of $H^0(X, \Omega_X)$.

The degree d-part of $\mathrm{Tor}_i^S(k, S_X)$ will be denoted by $\mathrm{Tor}_i^S(k, S_X)_d$, which is a vector space of dimension $\beta_{i,d}$. We can use our computation in order to show that all $\mathrm{Tor}_i^S(k, S_X)_d$ are acted on by the group G, but this also follows by Koszul cohomology, see [2]. Indeed, one starts with the vector space $V = H^0(X, \Omega_X)$, $\dim V = g$, $S = \mathrm{Sym}(V)$ and considers the exact Koszul complex

$$0 \to \wedge^g V \otimes S(-g) \to \wedge^{g-1} V \otimes S(-g+1) \to \cdots$$
$$\cdots \to \wedge^2 V \otimes S(-2) \to V \otimes S(-1) \to S \to k \to 0.$$

The symmetry property of the Tor functor implies that one can calculate $\mathrm{Tor}_i^S(k, S_X)$ by using the Koszul resolution of k, instead of the Koszul resolution of S_X. Since the Koszul resolution of k is a complex of G-modules and all differentials are G-module morphisms the $\mathrm{Tor}_i^S(k, S_X)_d$ are naturally G-modules. On the other hand the passage to the action on generators is not explicit, since the isomorphism between the graded components of the terms in the minimal resolution and Koszul cohomology spaces is not explicit, as as it comes from the spectral sequence that ensures the symmetry of Tor functor.

The representations to the d graded space of each F_i, $\rho_{i,d} : G \to \mathrm{GL}(F_{i,d})$ can be expressed as a direct sum of the G-modules $\mathrm{Tor}_i^S(k, S_X)_d$. The G-module structure of all F_i is determined by knowledge of the G-module structure of $H^0(X, \Omega_X)$ and the G-module structure of each $\mathrm{Tor}_i^S(k, S_X)$ for all $0 \leq i \leq g-2$.

7.1.1. *Petri's theorem and the canonical ideal.* Consider a complete non-singular non-hyperelliptic curve of genus $g \geq 3$ over an algebraically closed field K. Let Ω_X denote the sheaf of holomorphic differentials on X.

Theorem 12 (Noether-Enriques-Petri). *There is a short exact sequence*

$$0 \to I_X \to \mathrm{Sym} H^0(X, \Omega_X) \to \bigoplus_{n=0}^{\infty} H^0(X, \Omega_X^{\otimes n}) \to 0,$$

where I_X is generated by elements of degree 2 and 3. Also if X is not a non-singular quintic of genus 6 or X is not a trigonal curve, then I_X is generated by elements of degree 2.

For a proof of this theorem we refer to [63], [18]. The ideal I_X is called *the canonical ideal* and it is the homogeneous ideal of the embedded curve $X \to \mathbb{P}_k^{g-1}$. The automorphism group of the ambient space $\mathbb{P}^{g-1}$ is known to be $\mathrm{PGL}_g(k)$, [23, example 7.1.1 p. 151]. On the other hand, every automorphism of X is known to act on $H^0(X, \Omega_X)$ giving rise to a representation

$$\rho : G \to \mathrm{GL}(H^0(X, \Omega_X)),$$

which is known to be faithful, when X is not hyperelliptic and $p \neq 2$, see [32]. The representation ρ in turn gives rise to a series of representations

$$\rho_d : G \to \mathrm{GL}(S_d),$$

where S_d is the vector space of degree d polynomials in the ring $S := k[\omega_1, \ldots, \omega_g]$.

7.1.2. *The automorphism group as an algebraic set.* Let $X \subset \mathbb{P}^r$ be a projective algebraic set. Is it true that every automorphism $\sigma : X \to X$ comes as the restriction of an automorphism of the ambient projective space, that is by an element of $\mathrm{PGL}_k(r)$? For instance, such a criterion for complete intersections is explained in [36, sec. 2]. In the case of canonically embedded curves $X \subset \mathbb{P}^{g-1}$ it is clear that any automorphism $\sigma \in \mathrm{Aut}(X)$ acts also on $\mathbb{P}^{g-1} = \mathrm{Proj} H^0(X, \Omega_X)$. In this way we arrive at the following:

Lemma 13. *Every automorphism $\sigma \in \mathrm{Aut}(X)$ corresponds to an element in $\mathrm{PGL}_g(k)$ such that $\sigma(I_X) \subset I_X$ and every element in $\mathrm{PGL}_g(k)$ such that $\sigma(I_X) \subset I_X$ gives rise to an automorphism of X.*

Let $A_1, \ldots, A_r$ be a set of linear independent $g \times g$ matrices such that the $w^t A_i w$ $1 \leq i \leq r$ generate the canonical ideal, and $w^t = (w_1, \ldots, w_g)$ is a basis of the space of holomorphic differentials. By choosing an ordered basis of the vector space of symmetric $g \times g$ matrices we can represent any symmetric $g \times g$ matrix A as an element $\bar{A} \in k^{\frac{g(g+1)}{2}}$, that is

$$\bar{\cdot} : \text{Symmetric } g \times g \text{ matrices} \longrightarrow k^{\frac{g(g+1)}{2}}$$
$$A \longmapsto \bar{A}$$

We can now put together the r elements $\bar{A}_i$ as a $g(g+1)/2 \times r$ matrix $\left(\bar{A}_1 | \cdots | \bar{A}_r\right)$, which has full rank r, since $\{A_1, \ldots, A_r\}$ are assumed to be linear independent.

Proposition 14. *An element $\sigma = (\sigma_{ij}) \in \mathrm{GL}_g(k)$ induces an action on the curve X, if and only if the $g(g+1)/2 \times 2r$ matrix*

$$B(\sigma) = \left[\bar{A}_1, \ldots, \bar{A}_r, \overline{\sigma^t A_1 \sigma}, \ldots, \overline{\sigma^t A_r \sigma}\right]$$

has rank r.

7.2. **The canonical ideal of HKG-covers.** In [44] the canonical ideal of an HKG-curve X/k is computed using the method described in [10], which roughly states that in order to show that a set of quadratic differentials generates the canonical ideal, it suffices to show that the "initial terms" of the differentials generate a large enough subspace of the degree 2 part of the polynomial ring of symmetric differentials. A breakdown process of an HKG-curve into Artin-Schreier extensions as described in [27] and [43] is used. This article gives a new description of the generating elements of the tower of Artin-Schreier extensions. The symmetric Weierstrass semigroup H at the unique ramification point is used together with the explicit bases of polydifferentials based on the semigroup given in [27, proposition 42].

7.3. **Relative Syzygies.** In [10] the theory of the canonical ideal for relative curves is introduced for of the relative curve, and the example of Bertin-Mézard is used to illustrate this construction. The relative canonical map is introduced and an analogue of Petri's Theorem for the relative curve $\mathcal{X}/R$ is proved, by constructing a diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I_{\mathcal{X}_\eta} & \hookrightarrow & S_L := L[\omega_1, \ldots, \omega_g] & \xrightarrow{\phi_\eta} & \bigoplus_{n=0}^{\infty} H^0(\mathcal{X}_\eta, \Omega^{\otimes n}_{\mathcal{X}_\eta/L}) & \longrightarrow & 0 \\
 & & \uparrow \otimes_R L & & \uparrow \otimes_R L & & \uparrow \otimes_R L & & \\
0 & \longrightarrow & I_{\mathcal{X}} & \hookrightarrow & S_R := R[W_1, \ldots, W_g] & \xrightarrow{\phi} & \bigoplus_{n=0}^{\infty} H^0(\mathcal{X}, \Omega^{\otimes n}_{\mathcal{X}/R}) & \longrightarrow & 0 \\
 & & \downarrow \otimes_R R/\mathfrak{m} & & \downarrow \otimes_R R/\mathfrak{m} & & \downarrow \otimes_R R/\mathfrak{m} & & \\
0 & \longrightarrow & I_{\mathcal{X}_0} & \hookrightarrow & S_k := k[w_1, \ldots, w_g] & \xrightarrow{\phi_0} & \bigoplus_{n=0}^{\infty} H^0(\mathcal{X}_0, \Omega^{\otimes n}_{\mathcal{X}_0/k}) & \longrightarrow & 0
\end{array}$$

whose rows are exact and where each square is commutative. A Nakayama-type criterion that reduces the problem of finding a generating set for the relative canonical ideal $I_{\mathcal{X}}$ to finding compatible generating sets for the canonical ideals on the two fibres. It is proved that if G is a set of homogeneous polynomials in $I_{\mathcal{X}}$ such that $G \otimes_R L$ generates $I_{\mathcal{X}_\eta}$ and $G \otimes_R k$ generates $I_{\mathcal{X}_0}$ then G generates $I_{\mathcal{X}}$.

7.3.1. *Betti tables in special and generic fibres.* In [9] syzygies and Betti numbers of graded $R[w_1, \ldots, w_g]$-modules are studied; the standard setting in the bibliography [16], [61] is for R to be a field, while the treatment of [9] makes the more general assumption that the base is a discrete valuation ring. The results are applicable to finitely generated graded modules over polynomial rings over discrete valuation rings, but the motivating example was the relative canonical ring.

More precisely, let R be a discrete valuation ring with residue field k and quotient field L. We write S, S_k and S_L for the polynomial rings in g variables over R, k and L respectively, and consider a finitely generated graded S-module M which is flat as an R-module. Flatness over R does imply that the generic fiber $M \otimes S_L$ and the special fiber $M \otimes S_k$ have the same Hilbert polynomial, however the Betti tables need not be the same. It is well known that, even in the case of monomial ideals, the minimal free resolution depends on the characteristic of the ground field, the classical example being the triangulation of the projective plane.

The syzygies and the Betti numbers of the S_L-module $\widehat{M}$ and the S_k-module $\overline{M}$ are related. The study of syzygies becomes automatically more challenging over S since the non-zero elements of the base PID R may not be invertible and modules might have torsion, see [1, chap. 4] for a more comprehensive account see also [64]. On the other hand simplicial homology over $\mathbb{Z}$ has been extensively studied and techniques have been developed to account for that case and the different behavior over $\mathbb{Q}$, [15]. The main result of [9] is the following theorem, where the t_{ij} can be explicit computed using the Smith normal form.

Theorem 15. *Let M be a finitely generated graded S-module which is flat as an R-module, Π_i be the i-th syzygy of M and $t_{i,j}$ be the number of nonzero cyclic summands of $\Pi_{i,j} \otimes R$, for $i \geq 0$. Then*
(1) $\beta_{i,j}(M) = \beta_{i,j}(\overline{M})$, for $i \geq 0$.
(2) $\beta_{i,j}(M) = \beta_{i,j}(\widehat{M}) + t_{i,j} + t_{i-1,j}$ for $i \geq 1$.

7.4. **Deformation functors of representations.** The deformation theory of representations of the general linear group is a classical topic of study, see [49, prop. 1], [48, p.30], while deformation of representations in $\mathrm{Aut}\Gamma[[t]]$ corresponds to the local deformation functor in [12],[4].

The functors in these cases are given by

$$F : \mathrm{Ob}(\mathcal{C}) \ni \Gamma \mapsto \left\{ \begin{array}{l} \text{liftings of } \rho : G \to \mathrm{GL}_n(k) \\ \text{to } \rho_\Gamma : G \to \mathrm{GL}_n(\Gamma) \text{ modulo} \\ \text{conjugation by an element} \\ \text{of } \ker(\mathrm{GL}_n(\Gamma) \to \mathrm{GL}_n(k)) \end{array} \right\} \tag{7.1}$$

$$D_P : \mathrm{Ob}(\mathcal{C}) \ni \Gamma \mapsto \left\{ \begin{array}{l} \text{liftings of } \rho : G \to \mathrm{Aut}k[[t]] \\ \text{to } \rho_\Gamma : G \to \mathrm{Aut}\Gamma[[t]] \text{ modulo} \\ \text{conjugation by an element} \\ \text{of } \ker\left(\mathrm{Aut}\Gamma[[t]] \to \mathrm{Aut}k[[t]]\right) \end{array} \right\} \tag{7.2}$$

$\mathcal{G}(\Gamma)$	$\mathcal{R}(\Gamma)$	tangent space	action
$\mathrm{GL}_g(\Gamma)$	$\mathrm{End}_g(\Gamma)$	$\mathrm{End}_g(k) = M_g(k)$	$M \mapsto \mathrm{Ad}(\sigma)(M)$
$\mathrm{Aut}\Gamma[[t]]$	$\mathrm{End}(\Gamma[[t]])$	$k[[t]]\frac{d}{dt}$	$f(t)\frac{d}{dt} \longmapsto \mathrm{Ad}(\sigma)\left(f(t)\frac{d}{dt}\right)$

Table 1. Comparing the two group functors

This similarity was the motivation of [41], where both situations are unified. More precisely, let $\mathcal{G} : \mathcal{C} \to$ Groups be a group functor, see [14, ch. 2]. In this article, we will be mainly interested in two group functors. The first one, GL_g, will be represented by the by the group scheme $G_g = \Lambda[x_{11}, \ldots, x_{gg}, \det(x_{ij})^{-1}]$, that is $\mathrm{GL}_g(\Gamma) = \mathrm{Hom}_\Lambda(G_g, \Gamma)$. The second one is the group functor from the category of rings to the category of groups $\mathcal{N} : \Gamma \mapsto \Gamma[[t]]$.

We also assume that each group $\mathcal{G}(\Gamma)$ is embedded in the group of units of some ring $\mathcal{R}(\Gamma)$ depending functorially on Γ. This condition is asked since our argument requires us to be able to add together certain group elements. We also assume that the additive group of the ring $\mathcal{R}(\Gamma)$ has the structure of direct product Γ^I, while $\mathcal{R}(\Gamma) = \mathcal{R}(\Lambda) \otimes_\Lambda \Gamma$. Notice, that I might be an infinite set, but since all rings involved are Noetherian Γ^I is flat, see [46, 4F].

A representation of the finite group G in $\mathcal{G}(\Gamma)$ is a group homomorphism

$$\rho : G \to \mathcal{G}(\Gamma),$$

where Γ is a commutative ring, and a deformation functor F_ρ for any local Artin algebra Γ with maximal ideal $\mathfrak{m}_\Gamma$ in $\mathcal{C}$ to the category of sets is defined as follows:

$$F_\rho : \Gamma \in \mathrm{Ob}(\mathcal{C}) \mapsto \left\{ \begin{array}{l} \text{liftings of } \rho : G \to \mathcal{G}(k) \\ \text{to } \rho_\Gamma : G \to \mathcal{G}(\Gamma) \text{ modulo} \\ \text{conjugation by an element} \\ \text{of } \ker(\mathcal{G}(\Gamma) \to \mathcal{G}(k)) \end{array} \right\} \tag{7.3}$$

A comparison of these tow functors is given in terms of Petri's theorem and the canonical ideal and a method is given in order to reduce the complex deformation problem of curves with automorphisms to a better-understood problem involving linear representations.

Unlike previous approaches that relied on the subtle and difficult automorphism group of formal power series (e.g., $\mathrm{Aut}\, k[[t]]$), this article uses the general linear group (GL). The deformation of the curve is studied through its canonical embedding $X \to \mathbb{P}^{g-1}$, allowing us to replace abstract cohomological constructions with linear algebra and matrix computations.

A major result is [41, Th. 4], which provides a criterion for when an automorphism of a curve can be "lifted" to its deformation:

Theorem 16. *An automorphism σ of a curve X can be lifted to a deformation X_A if and only if the canonical ideal I_{X_A} remains invariant under the action of σ. For small extensions of Artin rings, this lifting is possible only if the linear canonical representation of the group G also lifts and the resulting ideal remains invariant under that lifted action.*

The article identifies specific conditions and obstructions related to these liftings:

- Obstruction to Lifting: It is shown that the lifting of a representation is a "strong condition" and provide examples of representations in positive characteristic that cannot be lifted to characteristic zero.
- Compatibility Condition: A specific relationship is established between the cocycles derived from the deformation of the curve and those from the deformation of its linear representations.
- Rank Criterion: A practical way to check if an action can be infinitesimaly lifted by examining the rank of a specific matrix ($F_{\Gamma'}(\sigma)$) constructed from the coordinates of the quadratic generators of the canonical ideal.

7.5. **The generalized Oort conjecture.** In [40] the lifting problem for modular representations of metacyclic groups of the form $G = C_q \times C_m$ is investigated.

The group G, which is defined by generators σ (order m) and τ (order $q = p^h$) with the following relation: $\sigma\tau\sigma^{-1} = \tau^\alpha$ for some integer α. The order of α must satisfy $\alpha^m \equiv 1 \pmod q$. The prime-to-$p$ part, C_m, has an order m that is coprime to p. The core of the paper is Theorem 1, which provides a necessary and sufficient condition for a modular $k[G]$-module M to lift to an $R[G]$-module.

If the module M decomposes into a direct sum of indecomposable modules $V_\alpha(\epsilon_i, \kappa_i)$ (where ϵ relates to the action of σ and κ is the dimension), it lifts if and only if these modules can be partitioned into sets I_ν such that:

- Capacity: The total dimension of each set of modules does not exceed q.
- Eigenvalue Consistency: The total dimension of each set must be 0 or $1 \pmod m$.
- Action Compatibility: The modules within each set must have "compatible" actions of σ. Specifically, the eigenvalues must follow a "dynamical system" sequence where $\epsilon_{\sigma(i+1)} = \epsilon_{\sigma(i)}\alpha^{\kappa_{\sigma(i)}}$.

Methodology: The "Dynamical System" Approach. A version of the Jordan normal form is developed for endomorphisms of order p^h over a local ring R. They show that the action of the group generator σ on an initial basis element acts as an "initial condition". This condition then recursively determines the action on the rest of the module through a "dynamical system".

Context and Applications Local Oort Conjecture: The research is motivated by the "local lifting problem," which deals with lifting group actions on formal power series rings. This is connected to the proof of the Oort conjecture by F. Pop.

These results are applied in [39] in order to show that certain dihedral groups can not lift in characteristic zero.

References

[1] William W. Adams and Philippe Loustaunau. *An introduction to Gröbner bases*, volume 3 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1994.

[2] Marian Aprodu and Jan Nagel. *Koszul cohomology and algebraic geometry*, volume 52 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2010.

[3] José Bertin. Obstructions locales au relèvement de revêtements galoisiens de courbes lisses. *C. R. Acad. Sci. Paris Sér. I Math.*, 326(1):55–58, 1998.

[4] José Bertin and Ariane Mézard. Déformations formelles des revêtements sauvagement ramifiés de courbes algébriques. *Invent. Math.*, 141(1):195–238, 2000.

[5] Niels Borne. Cohomology of G-sheaves in positive characteristic. *Adv. Math.*, 201(2):454–515, 2006.

[6] Helmut Boseck. Zur Theorie der Weierstrasspunkte. *Math. Nachr.*, 19:29–63, 1958.

[7] Irene I. Bouw and Stefan Wewers. The local lifting problem for dihedral groups. *Duke Math. J.*, 134(3):421–452, 2006.

[8] Louis Hugo Brewis and Stefan Wewers. Artin characters, Hurwitz trees and the lifting problem. *Math. Ann.*, 345(3):711–730, 2009.

[9] Hara Charalambous, Kostas Karagiannis, Sotiris Karanikolopoulos, and Aristides Kontogeorgis. Syzygies of ideals of polynomial rings over principal ideal domains. In *Proceedings of the 45th International Symposium on Symbolic and Algebraic Computation*, ISSAC '20, page 78–82, New York, NY, USA, 2020. Association for Computing Machinery.

[10] Hara Charalambous, Kostas Karagiannis, and Aristides Kontogeorgis. The relative canonical ideal of the Artin-Schreier-Kummer-Witt family of curves. *Annales de l'institut Fourier*, 73(3):1085–1113, 2023.

[11] Ted Chinburg, Robert Guralnick, and David Harbater. The local lifting problem for actions of finite groups on curves. *Ann. Sci. Éc. Norm. Supér. (4)*, 44(4):537–605, 2011.

[12] Gunther Cornelissen and Fumiharu Kato. Equivariant deformation of Mumford curves and of ordinary curves in positive characteristic. *Duke Math. J.*, 116(3):431–470, 2003.

[13] Huy Dang, Soumyadip Das, Kostas Karagiannis, Andrew Obus, and Vaidehee Thatte. Local Oort groups and the isolated differential data criterion. *J. Théor. Nombres Bordeaux*, 34(1):251–269, 2022.

[14] Michel Demazure. *Lectures on p-divisible groups*, volume 302 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986. Reprint of the 1972 original.

[15] Jean-Guillaume Dumas, Frank Heckenbach, David Saunders, and Volkmar Welker. Computing simplicial homology based on efficient Smith normal form algorithms. In *Algebra, geometry, and software systems*, pages 177–206. Springer, Berlin, 2003.

[16] David Eisenbud. *The geometry of syzygies*, volume 229 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2005. A second course in commutative algebra and algebraic geometry.

[17] Barry Green and Michel Matignon. Liftings of Galois covers of smooth curves. *Compositio Math.*, 113(3):237–272, 1998.

[18] Mark Green and Robert Lazarsfeld. A simple proof of Petri's theorem on canonical curves. In *Geometry today (Rome, 1984)*, volume 60 of *Progr. Math.*, pages 129–142. Birkhäuser Boston, Boston, MA, 1985.

[19] Alexander Grothendieck. Sur quelques points d'algèbre homologique. *Tôhoku Math. J. (2)*, 9:119–221, 1957.

[20] Alexander Grothendieck. *Fondements de la géométrie algébrique. [Extraits du Séminaire Bourbaki, 1957–1962.]*. Secrétariat mathématique, Paris, 1962.

[21] Alexander Grothendieck. *Revêtements étales et groupe fondamental. Fasc. I: Exposés 1 à 5*, volume 1960/61 of *Séminaire de Géométrie Algébrique*. Institut des Hautes Études Scientifiques, Paris, 1963.

[22] David Harbater. Moduli of p-covers of curves. *Comm. Algebra*, 8(12):1095–1122, 1980.

[23] Robin Hartshorne. *Algebraic Geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[24] Robin Hartshorne. *Deformation theory*, volume 257 of *Graduate Texts in Mathematics*. Springer, New York, 2010.

[25] Nathan Jacobson. *Basic algebra. II*. W. H. Freeman and Company, New York, second edition, 1989.

[26] Sotiris Karanikolopoulos and Aristides Kontogeorgis. Integral representations of cyclic groups acting on relative holomorphic differentials of deformations of curves with automorphisms. *Proc. Amer. Math. Soc.*, 142(7):2369–2383, 2014.

[27] Sotiris Karanikolopoulos and Aristides Kontogeorgis. Automorphisms of curves and Weierstrass semigroups for Harbater-Katz-Gabber covers. *Trans. Amer. Math. Soc.*, 371(9):6377–6402, 2019.

[28] Kazuya Kato. Vanishing cycles, ramification of valuations, and class field theory. *Duke Math. J.*, 55(3):629–659, 1987.

[29] Nicholas M. Katz. Local-to-global extensions of representations of fundamental groups. *Ann. Inst. Fourier (Grenoble)*, 36(4):69–106, 1986.

[30] Bernhard Köck. Galois structure of Zariski cohomology for weakly ramified covers of curves. *Amer. J. Math.*, 126(5):1085–1107, 2004.

[31] Bernhard Köck and Aristides Kontogeorgis. Quadratic differentials and equivariant deformation theory of curves. *Ann. Inst. Fourier (Grenoble)*, 62(3):1015–1043, 2012.

[32] Bernhard Köck and Joseph Tait. Faithfulness of actions on Riemann-Roch spaces. *Canad. J. Math.*, 67(4):848–869, 2015.

[33] Kunihiko Kodaira and Donald C. Spencer. On deformations of complex analytic structures, i. *Annals of Mathematics*, 67(2):328–466, 1958.

[34] Kunihiko Kodaira and Donald C. Spencer. On deformations of complex analytic structures, ii. *Annals of Mathematics*, 71(1):43–76, 1960.

[35] A. Kontogeorgis. The ramification sequence for a fixed point of an automorphism of a curve and the Weierstrass gap sequence. *Math. Z.*, 259(3):471–479, 2008.

[36] Aristides Kontogeorgis. Automorphisms of Fermat-like varieties. *Manuscripta Math.*, 107(2):187–205, February 2002.

[37] Aristides Kontogeorgis. On the tangent space of the deformation functor of curves with automorphisms. *Algebra Number Theory*, 1(2):119–161, 2007.

[38] Aristides Kontogeorgis. Polydifferentials and the deformation functor of curves with automorphisms. *Journal of Pure and Applied Algebra*, 210(2):551–558, 2007.

[39] Aristides Kontogeorgis and Alexios Terezakis. An obstruction to the local lifting problem, 2023.

[40] Aristides Kontogeorgis and Alexios Terezakis. On the lifting problem of representations of a metacyclic group. *J. Algebra*, 659:745–779, 2024.

[41] Aristides Kontogeorgis and Alexios Terezakis. The canonical ideal and the deformation theory of curves with automorphisms. *J. Pure Appl. Algebra*, 229(8):Paper No. 108002, 32, 2025.

[42] Aristides Kontogeorgis, Alexios Terezakis, and Ioannis Tsouknidas. Automorphisms and the Canonical Ideal. *Mediterr. J. Math.*, 18(6):Paper No. 261, 2021.

[43] Aristides Kontogeorgis and Ioannis Tsouknidas. A cohomological treatise of HKG-covers with applications to the Nottingham group. *J. Algebra*, 555:325–345, 2020.

[44] Aristides Kontogeorgis and Ioannis Tsouknidas. A generating set for the canonical ideal of HKG-curves. *Res. Number Theory*, 7(1):Paper No. 4, 16, 2021.

[45] Masatake Kuranishi. On the locally complete families of complex analytic structures. *Annals of Mathematics*, 75(3):536–577, 1962.

[46] T. Y. Lam. *Lectures on modules and rings*, volume 189 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1999.
[47] Claus Lehr and Michel Matignon. Automorphism groups for p-cyclic covers of the affine line. *Compos. Math.*, 141(5):1213–1237, 2005.
[48] Alexander Lubotzky and Andy R. Magid. Varieties of representations of finitely generated groups. *Mem. Amer. Math. Soc.*, 58(336):xi+117, 1985.
[49] B. Mazur. Deformation theory of Galois representations (in Modular forms and Fermat's last theorem). pages xx+582, 1997. Papers from the Instructional Conference on Number Theory and Arithmetic Geometry held at Boston University, Boston, MA, August 9–18, 1995.
[50] Ian Morrison and Henry Pinkham. Galois Weierstrass points and Hurwitz characters. *Ann. of Math. (2)*, 124(3):591–625, 1986.
[51] Shōichi Nakajima. Action of an automorphism of order p on cohomology groups of an algebraic curve. *J. Pure Appl. Algebra*, 42(1):85–94, 1986.
[52] Shōichi Nakajima. On abelian automorphism groups of algebraic curves. *J. London Math. Soc. (2)*, 36(1):23–32, 1987.
[53] Shōichi Nakajima. p-ranks and automorphism groups of algebraic curves. *Trans. Amer. Math. Soc.*, 303(2):595–607, 1987.
[54] Louis Nirenberg. On elliptic partial differential equations. *Annali della Scuola Normale Superiore di Pisa*, 13:115–162, 1959.
[55] Andrew Obus. The (local) lifting problem for curves. In *Galois-Teichmüller theory and arithmetic geometry*, volume 63 of *Adv. Stud. Pure Math.*, pages 359–412. Math. Soc. Japan, Tokyo, 2012.
[56] Andrew Obus. The local lifting problem for A_4. *Algebra Number Theory*, 10(8):1683–1693, 2016.
[57] Andrew Obus. A generalization of the Oort conjecture. *Comment. Math. Helv.*, 92(3):551–620, 2017.
[58] Andrew Obus and Stefan Wewers. Cyclic extensions and the local lifting problem. *Ann. of Math. (2)*, 180(1):233–284, 2014.
[59] Frans Oort. Lifting algebraic curves, abelian varieties, and their endomorphisms to characteristic zero. In *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, volume 46 of *Proc. Sympos. Pure Math.*, pages 165–195. Amer. Math. Soc., Providence, RI, 1987.
[60] Guillaume Pagot. *Relèvement en caractéristique zéro d'actions de groupes abéliens de type* $(p, \dots, p)$. PhD thesis, Université Bordeaux I, available at http://www.math.u-bordeaux1.fr/~mmatigno/Pagot-These.pdf, 2002.
[61] Irena Peeva. *Graded syzygies*, volume 14 of *Algebra and Applications*. Springer-Verlag London, Ltd., London, 2011.
[62] Florian Pop. The Oort conjecture on lifting covers of curves. *Ann. of Math. (2)*, 180(1):285–322, 2014.
[63] B. Saint-Donat. On Petri's analysis of the linear system of quadrics through a canonical curve. *Math. Ann.*, 206:157–175, 1973.
[64] Mahrud Sayrafi. Computations over local rings in macaulay2, 2017.
[65] Michael Schlessinger. Functors of Artin rings. *Trans. Amer. Math. Soc.*, 130:208–222, 1968.
[66] T. Sekiguchi, F. Oort, and N. Suwa. On the deformation of Artin-Schreier to Kummer. *Ann. Sci. École Norm. Sup. (4)*, 22(3):345–375, 1989.
[67] Tsutomu Sekiguchi and Noriyuki Suwa. Théorie de Kummer-Artin-Schreier et applications. *J. Théor. Nombres Bordeaux*, 7(1):177–189, 1995. Les Dix-huitièmes Journées Arithmétiques (Bordeaux, 1993).
[68] Edoardo Sernesi. *Deformations of algebraic schemes*, volume 334 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2006.
[69] Jean-Pierre Serre. Sur la topologie des variétés algébriques en caractéristique p. In *Symposium internacional de topología algebraica International symposium on algebraic topology*, pages 24–53. Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958.
[70] Jean-Pierre Serre. Exemples de variétés projectives en caractéristique p non relevables en caractéristique zéro. *Proc. Nat. Acad. Sci. U.S.A.*, 47:108–109, 1961.
[71] Jean-Pierre Serre. *Local fields*. Springer-Verlag, New York, 1979. Translated from the French by Marvin Jay Greenberg.
[72] I. R. Shafarevich. Abelian and nonabelian mathematics. *Math. Intelligencer*, 13(1):67–75, 1991. Translated from the Russian by Smilka Zdravkovska.
[73] Henning Stichtenoth. *Algebraic function fields and codes*. Springer-Verlag, Berlin, 1993.
[74] Bradley Weaver. The local lifting problem for D_4. *Israel J. Math.*, 228(2):587–626, 2018.
[75] Stefan Wewers. Deformation of tame admissible covers of curves. In *Aspects of Galois theory (Gainesville, FL, 1996)*, volume 256 of *London Math. Soc. Lecture Note Ser.*, pages 239–282. Cambridge Univ. Press, Cambridge, 1999.
[76] Ernst Witt. Zyklische Körper und Algebren der Charakteristik p vom Grad p^n. Struktur diskret bewerteter perfekter Körper mit vollkommenem Restklassenkörper der Charakteristik p. *J. Reine Angew. Math.*, 176:126–140, 1937.