Braid groups, Galois groups and some algebraic curves

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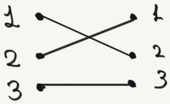
What is a Braid group?

$(12) \in S_3$



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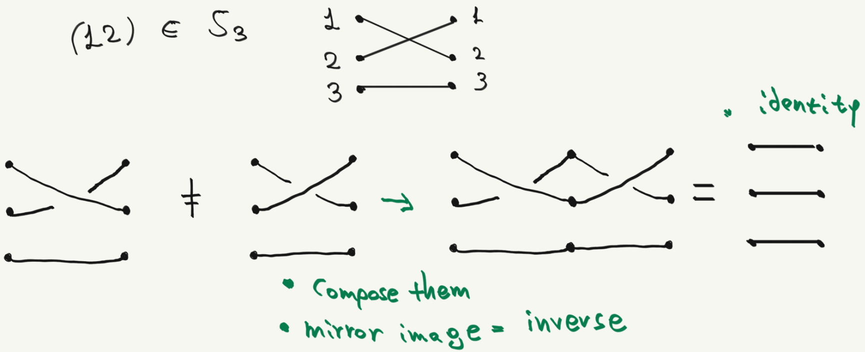

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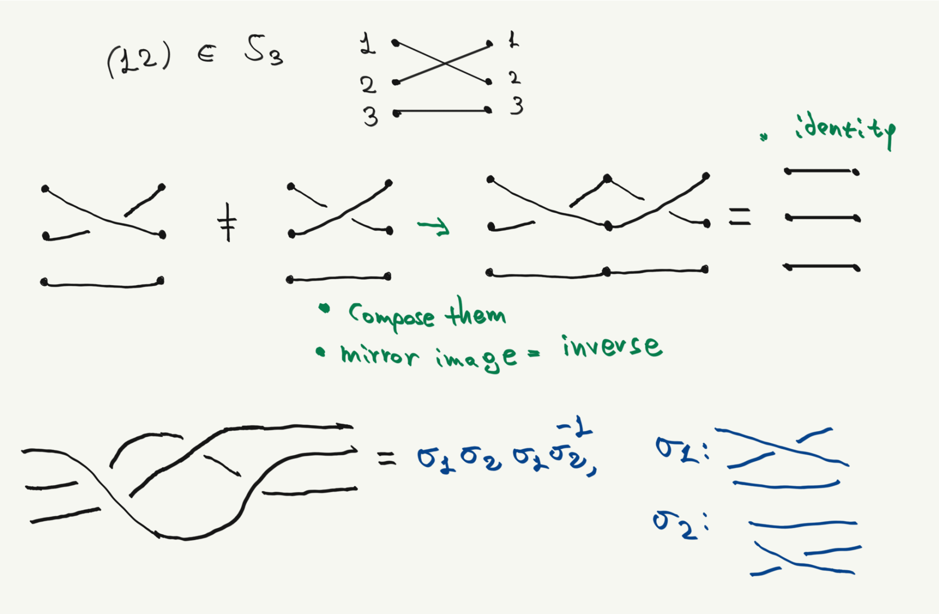

Artin representation of B_3

Let $F = \langle x_1, x_2, x_3 \mid x_1 x_2 x_3 = 1 \rangle$ be the free group on two generators x_1, x_2. Then,

$$B_3 \leq \mathrm{Aut}(F)$$

generated by σ_1, σ_2 such that

$$\sigma_i(x_{i+1}) = x_i,\ \sigma_i(x_i) = x_i x_{i+1} x_i^{-1},\ \sigma_i(x_k) = x_k$$

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Pure braids

In particular, there is a subgroup:

$$PB_3 = \{\sigma \in \mathrm{Aut}(F) : \sigma(x_i) \sim x_i\},$$

where $\sim$ denotes conjugation by an element (differs for every input).

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A wild Modular group appeared!

$$B_3/Z(B_3) \cong \mathrm{SL}_2(\mathbb{Z})/\{\pm I\},$$

where $Z(B_3)$ is the center generated by $(\sigma_1\sigma_2)^3$. Remember $(ST)^3 = 1$?

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A scary generalization

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Examples

$\mathrm{Mod}(D_3) = B_3$, where D_3 is the closed disk with three marked points.
$\mathrm{Mod}(S_{0,4}) = \mathrm{PSL}_2(\mathbb{Z})$, $\quad S_{0,4}$ is the sphere with 4 punctures.
$\mathrm{Mod}(S)$ is a subgroup of $\mathrm{Out}(\pi_1(S))$ in general.

Galois Theory reminder

Galois group

$$\mathrm{Gal}(K/L) = \{\sigma : K \to K, \text{ field automorphisms }, \ \sigma(x) = x \text{ for all } x \in L\}$$

$$\begin{array}{ccc}
K = \mathbb{Q}(\sqrt{3}+\sqrt{2}+\sqrt{5}) & & 1 \\
| & & | \\
L_1 = \mathbb{Q}(\sqrt{3}, \sqrt{2}) & & \mathrm{Gal}(K/L_1) \\
| & & | \\
L_2 = \mathbb{Q}(\sqrt{3}) & & \mathrm{Gal}(K/L_2) \\
| & & | \\
\mathbb{Q} & & \mathrm{Gal}(K/\mathbb{Q})
\end{array}$$

What is the Fundamental group $\pi_1(X)$?

Quick answer: A Galois group (of some sort...)

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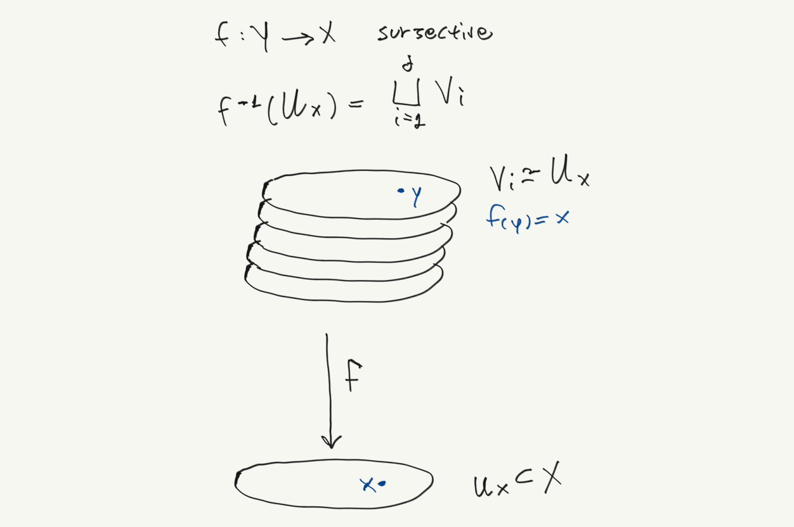
$\pi_1(X)$ classifies:



Examples

$$p : \mathbb{R} \longrightarrow S^1$$
$$\theta \longmapsto e^{i\theta},$$

and $\pi_1(S^1) = \mathbb{Z}$.

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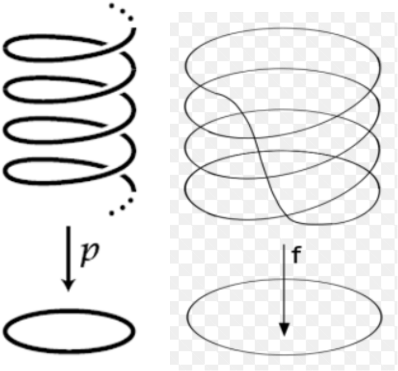
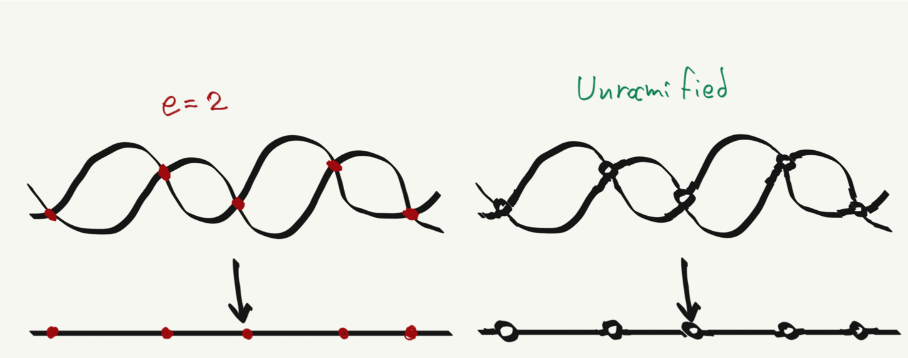
$$\theta \longmapsto e^{i\theta},$$

and $\pi_1(S^1) = \mathbb{Z}$. There is a subcovering f corresponding to $4\mathbb{Z}$:



Branched covers to topological covers



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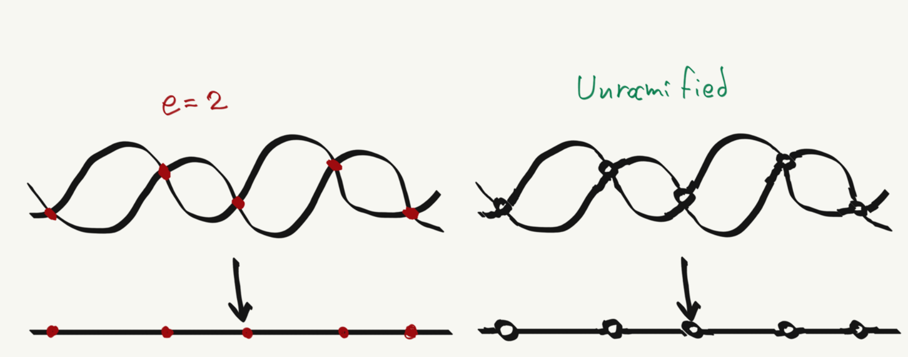

From elliptic curves

Let $E : y^2 = x(x-1)(x-\lambda)$ then the rational map to $\mathbb{P}^1$ being the first projection is ramified over $0, 1, \lambda, \infty$ with ramification indices 2.

P-adic numbers reminder

Profinite (or pro-p) completion

Given a group G, one can form the inverse limit

$$\varprojlim G/N$$

that runs over all finite index normal subgroups N with some sort of "compatible" maps between the quotients.

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$$\mathbb{Z}_p := \varprojlim \mathbb{Z}/p^n\mathbb{Z}$$

Two ways to view these:

$$x = \sum_{k=0}^{\infty} a_k p^k$$

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$$(\beta_n \bmod p^n) : \quad \beta_1 = a_0, \beta_2 = a_0 + a_1 p, \beta_3 = a_0 + a_1 p + a_2 p^2, \ldots$$

Two useful examples

Tate module of Elliptic Curve

$$\varprojlim E[p^n]$$

with compatible maps

$$E[p^{m+1}] \longrightarrow E[p^m]$$

can also be applied to the Weil pairing.

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Free pro-p group

The pro-p completion of F will be denoted by $\mathcal{F}$. Elements can be words of the form:

$$1, w, w^2, w^3, \ldots, w^x, \ldots$$

with $x = \sum_{k=0}^{\infty} a_k p^k \in \mathbb{Z}_p$ by some sort of "continuity".

Our favourite topological space...

... is the punctured projective line

Let X be $\mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\}$. Then $\pi_1(X) = F$

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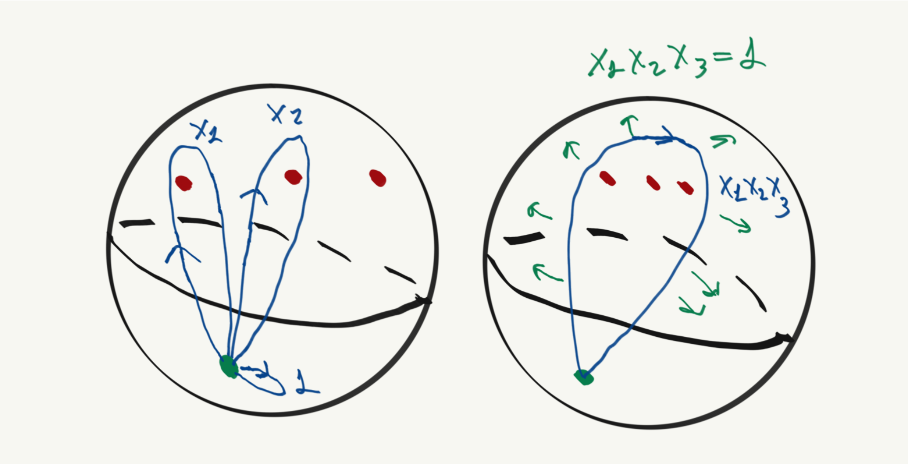

But why?

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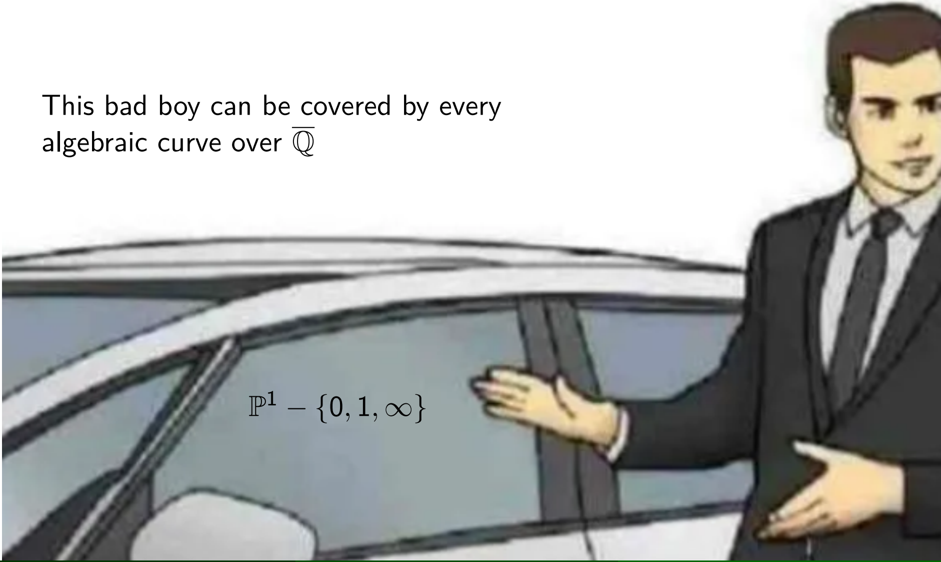
This bad boy can be covered by every algebraic curve over $\overline{\mathbb{Q}}$



A celebrated theorem

Theorem (Belyi)

The algebraic curve C over $\mathbb{C}$ can be defined over $\overline{\mathbb{Q}}$ if and only if there exists a ramified cover $C \to \mathbb{P}^1$ branched above three points.

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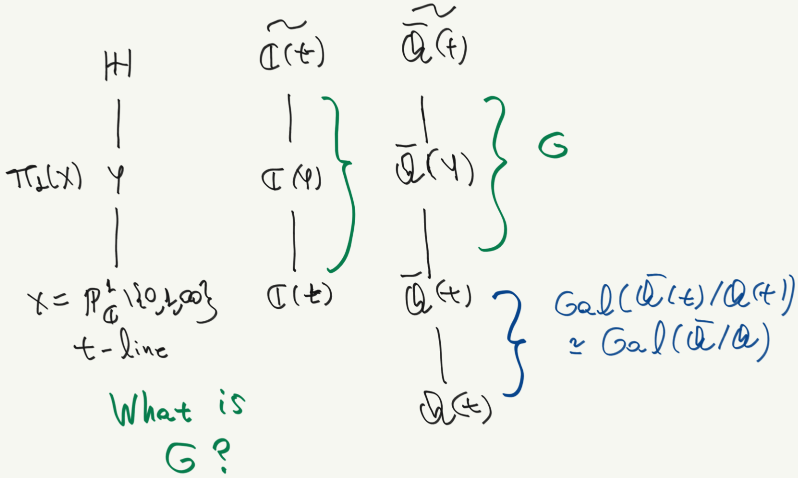
EC example revisited

For the elliptic curve $y^2 = x(x-1)(x-\lambda), \lambda \in \overline{\mathbb{Q}}$ the projection $(x, y) \mapsto x$ is ramified over $\{0, 1, \infty, \lambda\}$. Compose this with

$$\frac{x-\lambda}{x(1-\lambda)},$$

to reduce them to $\{0, 1, \infty\}$.

What are the finite covers over $\mathbb{Q}$?



Throw everything to the mix

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The outer representation

$$1 \longrightarrow \mathcal{F} \longrightarrow \Pi_{\mathbb{Q}} \longrightarrow \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow 1$$

yields

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{Out}(\mathcal{F})$$

with image

$$\{\sigma \in \mathrm{Out}(\mathcal{F}) : \quad \sigma(x_i) \sim x_i^{a_\sigma}\}$$

for some p-adic number a_σ. In particular there is a subgroup for all σ such that $a_\sigma = 1$, familiar?

A climb up the Fermat tower

Through Galois correspondence, the commutator subgroup $\mathcal{F}' = [\mathcal{F}, \mathcal{F}]$ contains all fields:

$$\overline{\mathbb{Q}}(t^{1/p^n}, (1-t)^{1/p^n}),$$

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Conjecture

Is $F_\sigma(\zeta_{p^n}^a - 1, \zeta_{p^n}^b - 1, \zeta_{p^n}^c - 1)$, well-defined?

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Is $F_\sigma(\zeta_{p^n}^a - 1, \zeta_{p^n}^b - 1, \zeta_{p^n}^c - 1)$, well-defined? Answer: Yes! (Ihara-Kaneko-Yukinari, Anderson, Coleman)

Some references

Inspiration

Y. Ihara, *Profinite braids, Galois representations and complex multiplications*, Annals of Mathematics (1986).
Y. Ihara, *Braid groups, Galois groups and arithmetic functions.*

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Ihara viewpoint

A. Kontogeorgis, P. Paramantzoglou, Galois action on homology of generalized Fermat curves. (2019)

Where to now?

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- The story goes on: Cyclic covers, to elementary abelian covers, to non-abelian covers

$$\begin{aligned} y^n = f(x), &\quad \mathbb{Z}/n\mathbb{Z}, \\ x^n + y^n = z^n, &\quad \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}, \\ \text{Heisenberg curves}, &\quad (\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}) \rtimes \mathbb{Z}/n\mathbb{Z} \end{aligned}$$

"A climb up the Heisenberg tower", (in preparation, not soon... or maybe...)

Thank you!

Ευχαριστώ πολύ!

$$\downarrow$$

$$x^n + y^n = 1$$

$$\downarrow$$

$$\mathbb{P}^1 \setminus \{0, 1, \infty\}$$