GALOIS ACTION ON HOMOLOGY OF THE HEISENBERG CURVE.

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Abstract. The Heisenberg curve is defined topologically as a cover of the Fermat curve and corresponds to an extension of the projective line minus three points by the non-abelian Heisenberg group modulo n. We compute its fundamental group and investigate an action from Artin's Braid group to the curve itself and its homology. We also provide a description of the homology in terms of irreducible representations of the Heisenberg group over a field of characteristic 0.

1. Introduction

In [12], [11] the fundamental group of an open abelian Galois cover $X \to \mathbb{P}^1$ of the projective line was computed and used in order to study the actions of the automorphism group $\mathrm{Aut}(X)$, the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and the braid group on the homology of the curve X. In this article we follow the same approach in order to study the Heisenberg curve, which is a Galois group of the projective line with Galois group the non-abelian discrete Heisenberg group H_n, see definition 7. The automorphism of the Heisenberg curve was studied in [1].

The homology group as a $\mathbb{F}[H_n]$-module, over a field $\mathbb{F}$ of characteristic zero, containing the n-th roots of unity, is given in theorem 27

$$H_1(X_H, \mathbb{F}) = \bigoplus_{j=0}^{n-1} \bigoplus_{i,s=0}^{\gcd(n,j)-1} \mathbb{F} h_{ijs} \chi_{ijs},$$

where the coefficients $h_{ijs} \in \mathbb{Z}$ are explicitly described in eq. (12) and the irreducible characters χ_{ijs} are discussed in the appendix.

The approach we use is as follows. From the ramified cover $X \to \mathbb{P}^1$ we can remove the branched points $0, 1, \infty$ and obtain an open cover $X^0 \to X_3 := \mathbb{P}^1 \setminus \{0, 1, \infty\}$. Covering space theory then provides that the open curve X^0 can be described as a quotient of the universal covering space $\tilde{X}_3$ by the fundamental group of the open curve $\pi_1(X^0, x') < \pi_1(X_3, x)$ for a fixed point $x \in X_3$ and a randomly chosen fixed preimage x' in X^0. The fundamental group $\pi_1(X_3, x)$ is the free group F_2 of rank 2 and can be presented as

$$\pi_1(X_3, x) = \langle x_1, x_2, x_3 | x_1 x_2 x_3 = 1 \rangle,$$

with each x_i representing the homotopy class of loops on x around the punctures $0, 1, \infty$ respectively.

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This setting fits the framework of Y. Ihara as in [9], [10]. Firstly, the braid group B_3 can be realized as a subgroup of $\mathrm{Aut}(F_2)$ generated by the elements σ_i for $i = 1, 2$ given by

$$\sigma_i(x_k) = \begin{cases} x_k & k \neq i, i+1, \\ x_i x_{i+1} x_i^{-1} & k = i, \\ x_i & k = i+1. \end{cases}$$

where we use that $x_3 = (x_1 x_2)^{-1}$. There is a natural surjection $B_3 \to S_3$ with its kernel being the so called pure braid group P_n, where every element $\sigma \in P_n$, satisfies

$$\sigma(x_k) \sim x_k^{N(\sigma)}, \ \text{ for some } N(\sigma) \in \mathbb{Z}^* = \{\pm 1\},$$

for $N(\sigma)$ not depending on x_k and by $\sim$ we denote conjugation. According to Ihara the pure braid group is a discrete analogue of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ in the following sense. Fixing a prime ℓ, by considering the étale pro-ℓ fundamental group of $\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{P_1, P_2, \infty\}$, with $P_i \in \mathbb{Q}$, he introduced the monodromy representation

$$\mathrm{Ih}_2 : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{Aut}(\mathfrak{F}_2),$$

with $\mathfrak{F}_2$ being the pro-ℓ completion of F_2. The group $\mathfrak{F}_2$ can be a considered as a quotient in the pro-ℓ category of the free pro-ℓ group $\mathfrak{F}_3$ and admit a similar presentation $\mathfrak{F}_2 = \langle x_1, x_2, x_3 | x_1 x_2 x_3 = 1 \rangle$. Ihara's representation has image inside the group

$$\left\{ \sigma \in \mathrm{Aut}(\mathfrak{F}_2) : \ \sigma(x_i) \sim x_i^{N(\sigma)}, (1 \leq i \leq 3) \text{ for some } N(\sigma) \in \mathbb{Z}_\ell^* \right\}, \tag{1}$$

where again $N(\sigma)$ does not depend on x_i and the composition $N \circ \mathrm{Ih}_2 : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}_\ell^*$ coincides with the cyclotomic character χ_ℓ.

We begin by considering the perhaps most famous curve in number theory, that is the Fermat curve in projective coordinates $x^n + y^n = z^n$, which has a fundamental group denoted by $R_{\mathrm{F}er_n}$. By considering a subgroup of $R_{\mathrm{F}er_n}$ that is the kernel of the surjection $F_2 \to H_n$ we obtain the Heisenberg curve through the covering subspaces Galois correspondence of $\pi_1(X_3, x)$. This kernel denoted by $R_{\mathrm{H}eis_n}$ is precisely the fundamental group of the Heisenberg curve. We then use the free generators of $R_{\mathrm{F}er}$ as stepping stones and employ the Schreier's lemma on them to compute the free generators of $R_{\mathrm{H}eis}$, in order to investigate the braid group action.

What happens is also interesting in the field of moduli versus field of definition point of view as in the work of Debes, Douai in [5] and Debes, Emsalem in [6]. Let K be a field and K_S a separable closure, then a curve defined a priori on K_S might not always be definable over K. Their work provide a cohomological measure to when this is possible, considering reductions to covers and their automorphisms while descending from K_S to K.

In that framework, similar to how Ihara derives the monodromy representation, an action of $\mathrm{Gal}(K_S/K)$ is lifted on the geometric fundamental group $\Pi_{K_S}(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \simeq \mathfrak{F}_2$ via the exact sequence

$$1 \longrightarrow \Pi_{K_S}(\mathbb{P}^1 \setminus \{0,1,\infty\}) \longrightarrow \Pi_K(\mathbb{P}^1 \setminus \{0,1,\infty\}) \longrightarrow \mathrm{Gal}(K_S/K) \to 1,$$

which induces an action on the covers of $\mathbb{P}^1$. The field of moduli for a cover X is defined to be the fixed field of the automorphisms in $\mathrm{Gal}(K_S/K)$ that produce a cover isomorphic to X.

In contrast to this arithmetic action on $\mathfrak{F}_2$, in this paper we consider the geometric in nature action of the braid group and its induced action on the covers. The Heisenberg curve turns out to be an interesting example in this setting, whereas the Fermat curve having a fundamental group that is a characteristic subgroup of F_2 stays invariant under the braid group action. We provide a case where the Heisenberg curve under the braid action gets mapped to an entirely new non-isomorphic curve.

Structure of the paper. In section 2 we provide the preliminaries in 2.1 and define the Heisenberg curve as a cover of the Fermat curve in 2.2. Then in 2.3 we compute its fundamental group and describe the Galois action in 2.3.1. Afterwards, in 2.4 we describe the elements that correspond to lifts of homotopy classes of loops around ∞ in terms of our previously established generators. Moreover, in 2.5 we discuss all the elements of lifts around the punctures and define the homology of the compactified curve after removing those points. In 2.6 we investigate the braid action on the curve and its homology and discuss the Burau representation in 2.6.1. Finally, in 3 we discuss the theory of Alexander Modules leading up to the proof of our main theorem. We provide an appendix A of the irreducible characters of H_n as many computations in section 3 rely on them and we also provide the following small case example, highlight many parts of the paper.

Example 1. For $n = 3$ the Fermat curve X_{F_3} is given by the affine equation $x^3 + y^3 = 1$ and it has a projective canonical weierstrass equation $zy^2 = x^3 - 432z^3$ with genus 1. The Heisenberg curve X_{H_3} is also of genus 1, thus we have an isogeny of elliptic curves $X_{H_3} \to X_{F_3}$ and an equation of X_{H_3} has been computed in [1] to be $y^2 = x^3 + 2^4 \cdot 3^6$.

In terms of group theory, the open Fermat and Heisenberg curves can be defined as quotients of the universal covering space $\tilde{X}_3$ of $X_3 = \mathbb{P}^1 \setminus \{0, 1, \infty\}$. More precisely, if $F_2 = \langle a, b\rangle$ is the Galois group of $\tilde{X}_3$ over X_3, then the open curves $X^\circ_{F_3}, X^\circ_{H_3}$ are defined by the fundamental groups

$$\langle a^3, b^3, [a, b]\rangle,\ \langle a^3, b^3, [a, [a, b]], [b, [a, b]]\rangle$$

respectively, as subgroups of F_2. Quotiening these by $\Gamma = \langle a^3, b^3, (ab)^3\rangle$ which are the third powers of the classes of loops around the branched points $\{0, 1, \infty\}$, we obtain the fundamental groups of the curves X_{F_3}, X_{H_3} which we expect to be isomorphic to $\mathbb{Z} \times \mathbb{Z}$. Indeed, in [11] the free generators of the fundamental group of the Fermat curve have been computed and for $n = 3$ the generators of $H_1(F_3, \mathbb{Z})$ are $[a, b] \bmod \Gamma$ and $[a, b]^a \bmod \Gamma$. We provide a SageMath [15] script[1] which verifies the word problem that $[a, b], [a, b]^a$ indeed commute $\bmod \Gamma$.

Let $T = [a, b]$ and denote by $x^y = yxy^{-1}$, using the results from this paper we compute that the free generators of the fundamental group of $X^\circ_{H_3}$ are

$$(a^3)^{b^iT^j},\ (b^3)^{a^iT^j}, \quad 0 \le i, j \le 2,$$
$$((ab)^3)^{a^iT^j}, \quad 0 \le i, j \le 2,\ (i, j) \ne (0, 0),$$
$$[a, T],\ [a, T]^a.$$

Thus, $H_1(X_{H_3}, \mathbb{Z})$ is generated by the classes of $[a, T]$ and $[a, T]^a$ modulo Γ and SageMath can verify that also these commute. Lastly, for a field $\mathbb{F}$ of characteristic 0

[1]The script can be found at `https://github.com/noulasd/HeisenbergCurve`

which contains the roots of x^3-1, if χ_{ijs} are the irreducible characters of the discrete Heisenberg group H_3 as described in the appendix, the regular representation is

$$\mathbb{F}[H_n] = \bigoplus_{i,s=0}^{2} \mathbb{F}\chi_{i0s} \oplus \mathbb{F}3\chi_{010} \oplus \mathbb{F}3\chi_{020},$$

of 9 one-dimensional and 2 three-dimensional irreducible representations. The homology over $\mathbb{F}$ as a subrepresentation has the character χ_{ijs} appearing $3/\gcd(3,j) - z_j(i,s)$ times, for $z_j(i,s)$ as defined in the Alexander Modules section, that is we recover

$$H_1(X_{H_3}, \mathbb{F}) = \mathbb{F}\chi_{101} \oplus \mathbb{F}\chi_{202},$$

as the two-dimensional vector space of the torus that corresponds to the elliptic curve.

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2. Heisenberg Curves

2.1. Group theory preliminaries.

Definition 2. The commutator $[x,y]$ of two elements x,y in a group is defined as $[x,y] = xyx^{-1}y^{-1}$.

Definition 3. The exponent x^y of two elements x,y in a group is defined as $x^y = xyx^{-1}$.

Definition 4. For a group G the subgroup G' is generated by the commutators $[g,h]$ for g,h in G. The abelizanization is defined as $G^{\mathrm{ab}} := G/G'$.

Lemma 5. *With the above definitions, for three elements x,y,z in a group, the following identity holds.*

$$[xy,z] = [y,z]^x \cdot [x,z].$$

Proof.

$$\begin{aligned}[xy,z] &= xyz \cdot y^{-1}x^{-1}z^{-1} = xyz \cdot (y^{-1}z^{-1}) \cdot (zy) \cdot y^{-1}x^{-1}z^{-1} \\ &= x[y,z]zx^{-1}z^{-1} = x[y,z]x^{-1}[x,z] = [y,z]^x \cdot [x,z].\end{aligned}$$

□

Lemma 6. *For two elements x,y in a group and for a positive integer j*

$$[x^j,y] = [x,y]^{x^{j-1}} \cdot [x,y]^{x^{j-2}} \cdots [x,y]^x \cdot [x,y],$$

$$[x,y^j] = [x,y] \cdot [x,y]^y \cdots [x,y]^{y^{j-2}} \cdot [x,y]^{y^{j-1}}.$$

Proof. See [7, 0.1, p.1] □

Definition 7. The (Discrete) Heisenberg group modulo n is defined as the group of matrices of the form

$$H_n := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad x, y, z \in \mathbb{Z}/n\mathbb{Z} \right\}.$$

It is generated by

$$a = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that,

$$[a, b] = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

According to [3], H_n admits a presentation:

$$H_n = \langle a, b \mid a^n, b^n,\ a[a, b] = [a, b]a,\ b[a, b] = [b, a]b \rangle.$$

It is evident from the matrices that we have the extra relation $[a, b]^n = 1$. We can derive it from the other relations as follows, since it will be useful later.

$$1 = a^n b^n = b \cdot a^n [a, b]^n b^{n-1} = b[a, b]^n b^{n-1} = [a, b]^n. \tag{2}$$

2.2. **The Heisenberg curve as a cover of the projective line and the Fermat curve.** In [1] the Heisenberg curve is defined as a cover of the projective line minus three points $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ with deck group H_n. The Heisenberg curve is a $\mathbb{Z}/n\mathbb{Z}$-cover of the Fermat curve, which in turn has deck group $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ over the punctured projective line. We denote as F_2 the free group $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, x_0)$ and by abusing notation we denote the generators by a and b, which are the classes of loops around 0 and 1 respectively. The whole data is depicted below in the following exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R_{\mathrm{Heis}_n} & \longrightarrow & F_2 & \longrightarrow & H_n & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & R_{\mathrm{Fer}_n} & \longrightarrow & F_2 & \longrightarrow & \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} & \longrightarrow & 0 \end{array}$$

where $R_{\mathrm{Heis}_n}, R_{\mathrm{Fer}_n}$ are the normal closures generated by the elements

$$R_{\mathrm{Fer}_n} := \langle a^n, b^n,\ [a, b] \rangle$$
$$R_{\mathrm{Heis}_n} := \langle a^n, b^n,\ a[a, b]a^{-1}[a, b]^{-1},\ b[a, b]b^{-1}[a, b]^{-1} \rangle$$

and $[a, b]^n \in R_{\mathrm{Heis}_n}$. We will omit the n and write R_{Heis} whenever it is clear in the given context, although because of the following results from [1] we will have to keep track of it at various times.

Lemma 8. *The Heisenberg curve is an unramified cover of the Fermat curve if n is odd and a ramified one if n is even. Moreover, in the ramified case the points above ∞ have ramification index $2n$.*

Proof. see [1, lemma 11] for the first statement and [1, proof of lemma 14] for the second statement. □

Using the Riemann-Hurwitz genus formula we obtain the following

Lemma 9. *The genus g of the (closed) Heisenberg curve is*

$$g = \begin{cases} \frac{n^2(n-3)}{2} + 1 & \text{if } \gcd(n,2) = 1, \\ \frac{n^2(n-3)}{2} + \frac{n^2}{4} + 1 & \text{if } 2 \mid n. \end{cases}$$

Proof. See [1, lemma 15]. □

2.3. **Fundamental group of the Heisenberg curve.** In this section we will describe the fundamental group of the Heisenberg curve. To do this, we will make use of the Schreier lemma [4, chap. 2, sec. 8] which given a free group and a generating set of it, it can provide a generating set for a given subgroup under some conditions.

More precisely, for a free group F with basis $X = \{x_1, \ldots, x_s\}$ a Schreier Transversal of a subgroup H is a set T of reduced words such that all initial segments of a word in T is also in T and for every coset of H in F contains a unique word of T. We will denote this unique word by $\overline{g}$ for every g in F. The Schreier's lemma concludes that H has a freely generating set consisting of the elements $\gamma(t,x) := tx\overline{tx}^{-1}, t \in T, x \in X$, whenever tx is not in T and $\gamma(t,x)$ does not reduce to 1.

We will use the description of the fundamental group of the Fermat curve $R_{\mathrm{F}er}$ as it is given in [11], and compute $R_{\mathrm{H}eis}$ using the Schreier's lemma. The group $R_{\mathrm{F}er}$ has the description

$$R_{\mathrm{F}er} = \langle a_1, b_1, \ldots, a_g, b_g, \gamma_1, \ldots, \gamma_{3n} \mid \gamma_1\gamma_2\cdots\gamma_{3n} \cdot [a_1,b_1][a_2,b_2]\cdots[a_g,b_g] = 1\rangle,$$

where $g = \frac{(n-1)(n-2)}{2}$ is the genus of the Fermat curve. The $3n$ elements γ_m are the elements $(a^n)^{b^i}, (b^n)^{a^i}$ and $((ab)^n)^{a^i}$ for $0 \le i \le n-1$ and the $2g$ elements remaining are $[b,a]^{a^i b^j}$ for $0 \le i \le n-2$ and $0 \le j \le n-3$. Under a base change in [11], which even though is written in additive notation only the group operation is being used, we have the free generators of the open Fermat curve:

$$(a^n)^{b^i}, (b^n)^{a^i},\ 0 \le i \le n-1,\ \ [a,b]^{a^i b^j},\ 0 \le i, j, \le n-2,$$

which will be our initial input in the Schreier process.

As the Heisenberg curve is a $\mathbb{Z}/n\mathbb{Z}$-cover of the Fermat curve, $R_{\mathrm{F}er_n}/R_{\mathrm{H}eis_n}$ is cyclic, generated by the commutator $T := [a,b]$. Therefore we choose $T^k, 0 \le k \le n-1$ as a Schreier transversal for $R_{\mathrm{F}er_n}/R_{\mathrm{H}eis_n}$. Before computing the generators in Schreier's algorithm, the following elementary lemma will be beneficial in capturing the information about the ramification throughout this paper.

Lemma 10. *In H_n we have that $(ab)^n = T^{\frac{n}{2}} = T^{-\frac{n}{2}}$ if $2 \mid n$, otherwise it is 1. Moreover, $(ab)^n$ is in $R_{\mathrm{H}eis_n}$ if and only if 2 does not divide n.*

Proof. In H_n the elements a and b commute with T, thus we can compute

$$\begin{aligned}(ab)^n &= Tb\underbrace{a^2b}_{\text{swap}}(ab)^{n-2} = T^{1+2}b^2\underbrace{a^3b}_{\text{swap}}(ab)^{n-3} = \cdots \\ &= T^{1+\cdots+(n-1)}b^{n-1}a^nb \\ &= T^{\frac{n(n-1)}{2}} \\ &= \begin{cases} 0, & \text{if } \gcd(n,2)=1 \\ T^{\frac{n}{2}}, & \text{if } 2 \mid n. \end{cases}\end{aligned}$$

For the second assesment, note that $H_n \cong F_2/R_{\mathrm{Heis}_n}$. □

Now we compute:

$$\begin{aligned}\overline{T^k(b^n)^{a^i}} &= T^k \\ \overline{T^k(a^n)^{b^i}} &= T^k \\ \overline{T^k[a,b]^{a^ib^j}} &= \begin{cases} T^{k+1}, & k \le n-1 \\ 1, & k = n-1 \end{cases}\end{aligned}$$

The above computations are straightforward, since they happen in H_n. We compute now the free generators:

$$T^k(b^n)^{a^i} \cdot \left(\overline{T^k(b^n)^{a^i}}\right)^{-1} = T^k(b^n)^{a^i}T^{-k}, \quad 0 \le i,k \le n-1$$

$$T^k(a^n)^{b^i} \cdot \left(\overline{T^k(a^n)^{b^i}}\right)^{-1} = T^k(a^n)^{b^i}T^{-k}, \quad 0 \le i,k \le n-1$$

$$T^k[a,b]^{a^ib^j} \cdot \left(\overline{T^k[a,b]^{a^ib^j}}\right)^{-1} = \begin{cases} T^{n-1}[a,b]^{a^ib^j}, & 0 \le i,j \le n-2, \\ & k = n-1 \\ T^k[a,b]^{a^ib^j}T^{-(k+1)}, & 0 \le k,i,j \le n-2, \\ & (i,j) \ne (0,0), \\ 1, & (i,j) = (0,0), 0 \le k \le n-2. \end{cases}$$

Lemma 11. *The generators of the free group R_{Heis_n} are listed below, as a union of the following sets:*

$$\begin{aligned} A_1 &= \{T^k(a^n)^{b^i}T^{-k}, \quad 0 \le i,k \le n-1\}, \quad \#A_1 = n^2, \\ A_2 &= \{T^k(b^n)^{a^i}T^{-k}, \quad 0 \le i,k \le n-1\}, \quad \#A_2 = n^2, \\ A_3 &= \{T^{n-1}T^{a^ib^j}, \quad 0 \le i,j \le n-2\}, \quad \#A_3 = (n-1)^2, \\ A_4 &= \{T^kT^{a^ib^j}T^{-(k+1)}, \quad 0 \le k,i,j \le n-2,\ (i,j) \ne (0,0)\}, \\ \#A_4 &= (n-1)^3 - (n-1). \end{aligned}$$

Proof. This is a direct consequence of the Schreier lemma. Notice that the above given sets add up to n^3+1 generators, as predicted by the Schreier index formula:

$$\begin{aligned} rank(R_{\mathrm{Heis}_n}) &= [F_2 : R_{\mathrm{Heis}_n}](2-1)+1 \\ &= [R_{\mathrm{Fer}_n} : R_{\mathrm{Heis}_n}](rank(R_{\mathrm{Fer}_n})-1)+1 \\ &= n^3+1 \end{aligned}$$

Indeed, we compute $\sum \#A_i = n^3+1$. $\square$

This basis will have a convenient form once Galois action is introduced, although it lacks ramification data. Specifically, we would like to be able to describe generators as homotopy classes of loops on the punctured tori R_{Heis} consists of, which would mean to have conjugates of $(ab)^n$ in the basis. Lemma 10 tells us that $(ab)^n$ will be in R_{Heis_n} for odd n and we can expect $(ab)^{2n}$ to be in R_{Heis_n} for even n. We will provide this in detail in a later section.

2.3.1. *Galois action.* Suppose we have a group G with a normal subgroup N, on which G acts by conjugation. We would like to define a conjugation action of G/N to N, induced from the action of G, but this can only be well-defined modulo inner automorphisms. Considering this problem, we can have a well-defined action of G/N on the abelianization N/N'. We will use this on the exact sequence

$$1 \to R_{\mathrm{Heis}} \to F_2 \to H_n \to 1$$

to have a Galois action on $R_{\mathrm{Heis}}^{\mathrm{ab}}$. Consider the two generators $\alpha = aR_{\mathrm{Heis}}$, $\beta = bR_{\mathrm{Heis}}$ as well as $\tau = [a,b]R_{\mathrm{Heis}}$ of the group H_n. Then, there exists a well-defined action of these three on $R_{\mathrm{Heis}_n}/R'_{\mathrm{Heis}_n}$ given by conjugation, that is

$$x^{\alpha} = x^{a} = axa^{-1}, \quad x^{\beta} = x^{b} = bxb^{-1}, \quad x^{\tau} = x^{T} = TxT^{-1}$$

for all $x \in R_{\mathrm{Heis}_n}/R'_{\mathrm{Heis}_n}$. Notice that this is an action, which implies that

$$(x^{\alpha})^{\tau} = x^{\tau\alpha} = x^{\alpha\tau} = (x^{\tau})^{\alpha}$$

$$\left(x^{\beta}\right)^{\tau} = x^{\tau\beta} = x^{\beta\tau} = (x^{\tau})^{\beta},$$

that is the actions of α and β commute with τ. In general, the actions of α and β do not commute in this case, as it happens in the Fermat curve. We will still use a, b, T as exponents to denote the conjugation as defined earlier and we use α, β, τ when the base of the exponent is in R_{Heis}, when we can realize this as an action from an element of H_n. For example, to make sense why this is necessary, we have that T is not in R_{Heis} and we can write $[a^n, T]$ as both $(a^n) - (a^n)^{\tau}$ and as $T^{a^n} \cdot T^{-1}$. Could the later one be realized as an action of H_n, the element would reduce to the identity.

Lemma 12. *The elements $T^n, T^{-n}[a^ib^j, T]$ are in R_{Heis}. In $R_{\mathrm{Heis}}/R'_{\mathrm{Heis}}$ they are decomposed as follows:*

$$T^n = a^n - (a^n)^\beta - \sum_{k=0}^{n-2}\sum_{i=0}^{n-2-k} [a,T]^{\tau^k \alpha^i} \tag{3}$$

$$T^{-n} = b^n - (b^n)^\alpha - \sum_{k=0}^{n-2}\sum_{j=0}^{n-2-k} [b,T^{-1}]^{\tau^{-k} \beta^j} \tag{4}$$

$$[a^ib^j, T] = [b,T]^{\alpha^i \sum_{\lambda=0}^{j-1} \beta^\lambda} + [a,T]^{\sum_{\lambda=0}^{i-1}\alpha^\lambda}. \tag{5}$$

Proof. The element T^n happens to be in A_3 for $(i,j) = (0,0)$, also we expected it to be in R_{Heis} because it is trivial in $H_n \cong F_2/R_{\mathrm{Heis}}$ from equation 2. Using lemma 5 and 6 we can write

$$[a^ib^j, T] = [b^j, T]^{a^i}[a^i, T] = [b,T]^{\alpha^i(\beta^{j-1}+\beta^{j-2}+\cdots+\beta+1)} \cdot [a,T]^{(\alpha^{i-1}+\alpha^{i-2}+\cdots+\alpha+1)},$$

which is in R_{Heis} and the decomposition follows in the abelianization. We also compute:

$$\begin{aligned}
a^n - (a^n)^\beta &= [a^n, b] = T^{a^{n-1}+\cdots+a+1} \\
&= [a^{n-1}, T]T \cdot [a^{n-2}, T]T \cdots [a,T]T \cdot T \\
&= [a^{n-1}, T]T(T^{-1}T) \cdot [a^{n-2}, T]T(T^{-2}T^2) \\
&\cdots [a,T]T(T^{-(n-1)}T^{n-1}) \cdot T(T^{-n}T^n) \\
&= [a^{n-1}, T] + [a^{n-2}, T]^\tau + \cdots + [a,T]^{\tau^{n-2}} + T^n
\end{aligned}$$

and the result follows. Similarly, we can prove eq. (4). ☐

Notice that in the second and third equality we have the group operation on elements of the form $[*, T]T$ and each T that is not inside the commutator contributes as a conjugation to every element that comes after it. As we will use this computation trick again, we will refer to it as *nested conjugations*. We rewrite now the sets A_i with H_n action:

$$\begin{aligned}
A_1 &= \{(a^n)^{\tau^k\beta^i}, \quad 0 \le i,k \le n-1\} \\
A_2 &= \{(b^n)^{\tau^k\alpha^i}, \quad 0 \le i,k \le n-1\} \\
A_3 &= \{T^n + [a^ib^j, T]^{\tau^{n-1}}, \quad 0 \le i,j \le n-2\} \\
A_4 &= \{[a^ib^j, T]^{\tau^k}, \quad 0 \le i,j,k \le n-2, (i,j) \ne (0,0)\}
\end{aligned}$$

and we can invertibly write our basis as $0 \le k \le n-1$:

$$(a^n)^{\tau^k\beta^i}, (b^n)^{\tau^k\alpha^i},\ 0 \le i \le n-1, \quad T^n, \quad [a^ib^j, T]^{\tau^k},\ 0 \le i,j \le n-2, (i,j) \ne (0,0).$$

Using lemma 12 we decompose in $R_{\mathrm{Heis}}/R'_{\mathrm{Heis}}$ even further:

$$\begin{aligned}
T^kT^{a^ib^j}T^{-(k+1)} &= [a^ib^j, T]^{\tau^k} = \\
&= [b,T]^{\tau^k\alpha^i \sum_{\lambda=0}^{j-1}\beta^\lambda} + [a,T]^{\tau^k \sum_{\lambda=0}^{i-1}\alpha^\lambda},
\end{aligned}$$

and

$$\begin{aligned} T^{n-1}T^{a^ib^j} &= T^nT^{-1}T^{a^ib^j}T^{-1}T = T^nT^{-1}[a^ib^j,T]T \\ &= T^n + [b,T]^{\tau^{n-1}\alpha^i\sum_{\lambda=0}^{j-1}\beta^\lambda} + [a,T]^{\tau^{n-1}\sum_{\lambda=0}^{i-1}\alpha^\lambda}. \end{aligned}$$

So far we have proved the following, which will be of use when investigating the braid action on the homology and the so called Burau representation.

Proposition 13. *For all n, the free $\mathbb{Z}[H_n]$-module R^{ab}_{Heis} is generated by the elements:*

$$a^n, b^n, [a,T], [b,T].$$

As a $\mathbb{Z}$-module, it can be generated by the n^3+1 elements: $0 \le k \le n-1$,

$$\begin{aligned} (a^n)^{\beta^i,\tau^k}, (b^n)^{\alpha^i\tau^k}, &\quad 0 \le i \le n-1, \\ [a,T]^{\alpha^i\tau^k}, &\quad 0 \le i \le n-3, (i,k) = (n-2,0), \\ [b,T]^{\alpha^i\beta^j\tau^k}, &\quad 0 \le i \le n-2, 0 \le j \le n-3. \end{aligned}$$

Proof. Follows by counting the indices that are possible to appear from the previous decomposition of $[a^ib^j,T]$. The element $[a,T]^{\alpha^{n-2}}$ follows from the decomposition of T^n. □

2.4. **Decomposition of $(ab)^n$.** The element $(ab)^n$ corresponds to the n-lift of the path in $\mathbb{P}^1 \setminus \{0,1,\infty\}$ around ∞, thus it is interesting to see how it decomposes in R_{Heis} in terms of our generators. From the ramification lemma 10 this will only be possible for odd n, and for even n we will be able to decompose in R_{Heis} the elements $(ab)^nT^{-\frac{n}{2}}, T^{\frac{n}{2}}(ab)^n$ from which we can form conjugates of $(ab)^{2n}$.

Lemma 14. *For all $n \in \mathbb{N}$, we can write the elements $(ab)^n$ as follows:*

$$(ab)^n = \prod_{i=1}^{n-1}\left(\prod_{j=0}^{i-1}[a^ib^{i-1-j},T^{-1}]T^{-1}\right) \cdot a^nb^n.$$

Proof. We begin by computing the $n=2$ case:

$$\begin{aligned} (ab)^2 &= abab^{-1}a^{-1}abb = a[b,a]a^{-1}aabb \\ &= a[b,a]a^{-1}aba^{-1}b^{-1}bab^{-1} \cdot abb \\ &= aT^{-1}a^{-1}Tbab^{-1} \cdot abb \\ &= [a,T^{-1}]T^{-1}a^2b^2, \end{aligned}$$

and the $n=3$ case:

$$\begin{aligned} (ab)^3 &= (ab)^2ab = [a,T^{-1}]T^{-1}a^2b^2ab \\ &= [a,T^{-1}]T^{-1}a^2b \cdot ba \cdot (b^{-1}a^{-1}ab) \cdot b \\ &= [a,T^{-1}]T^{-1}a^2bT^{-1} \cdot ab^2 \\ &= [a,T^{-1}]T^{-1}a^2bT^{-1} \cdot (b^{-1}a^{-2}TT^{-1}a^2b) \cdot ab^2 \\ &= [a,T^{-1}]T^{-1}[a^2b,T^{-1}]T^{-1} \cdot a^2bab^2 \end{aligned}$$

and

$$a^2bab^2 = a^2ba \cdot (b^{-1}a^{-1}a^{-2}TT^{-1}a^2ab) \cdot b^2 = [a^2, T^{-1}]T^{-1}a^3b^3,$$

thus

$$(ab)^3 = [a, T^{-1}]T^{-1} \cdot [a^2b, T^{-1}]T^{-1} \cdot [a^2, T^{-1}]T^{-1}a^3b^3,$$

and with the nested conjugations trick

$$(ab)^3 = [a, T^{-1}] \cdot [a^2b, T^{-1}]^{T^{-1}} \cdot [a^2, T^{-1}]^{T^{-2}} \cdot T^{-3} \cdot a^3b^3.$$

Let

$$E(k) := \prod_{i=1}^{k-1} \left(\prod_{j=0}^{i-1} [a^ib^{i-1-j}, T^{-1}]T^{-1} \right),$$

such that

$$(ab)^k = E(k)a^kb^k.$$

We prove the following claim, for positive integers μ, λ:

$$a^\mu b^\lambda ab = \prod_{j=0}^{\lambda-1} [a^\mu b^{\lambda-1-j}, T^{-1}]T^{-1}a^{\mu+1}b^{\lambda+1}.$$

Indeed:

$$\begin{aligned} a^\mu b^\lambda ab &= a^\mu b^{\lambda-1}ba(b^{-1}a^{-1}ab)b \\ &= a^\mu b^{\lambda-1}T^{-1}ab^2 \\ &= a^\mu b^{\lambda-1}T^{-1}(b^{-(\lambda-1)}a^{-\mu}TT^{-1}a^\mu b^{\lambda-1})ab^2 \\ &= [a^\mu b^{\lambda-1}, T^{-1}]T^{-1}a^\mu b^{\lambda-1}ab^2, \end{aligned}$$

and the result follows from recursion. Thus,

$$\begin{aligned} (ab)^{k+1} &= (ab)^k ab = E(k)a^kb^kab \\ &= E(k) \cdot \prod_{j=0}^{k-1} [a^kb^{k-1-j}, T^{-1}]T^{-1} \cdot a^{k+1}b^{k+1} \\ &= E(k+1)a^{k+1}b^{k+1}. \end{aligned}$$

□

We have written $(ab)^n$ into a succesion of elements of the form $[*, T^{-1}]T^{-1}$, so the nested conjugations trick will be applied. Notice that $T^{n(n-1)/2}$ is $\frac{n-1}{2}T^n$ in additive notation for odd n, but we cannot write this as an element in R_{Heis} for even n. Using the above and that $[*, T^{-1}] = -[*, T]^{\tau^{n-1}}$, we have in $R^{ab}_{\mathrm{Heis}_n}$ for odd n:

$$(ab)^n = -\sum_{i=1}^{n-1} \sum_{j=0}^{i-1} [a^ib^{i-1-j}, T]^{\tau^{\left(n-1-\frac{i(i-1)}{2}-j\right)}} + \frac{(n-1)}{2}T^{-n} + a^n + b^n,$$

and for even n we make a slight adjustment to the nested conjugations trick

$$(ab)^nT^{-\frac{n}{2}} = \prod_{i=1}^{n-1}\left(\prod_{j=0}^{i-1}[a^ib^{i-1-j},T^{-1}]T^{-1}\right)\cdot T^{\frac{n(n-1)}{2}}\cdot T^{-\frac{n(n-1)}{2}}\cdot T^{-\frac{n}{2}}T^{\frac{n}{2}}\cdot a^nb^n\cdot T^{-\frac{n}{2}}$$

$$= -\sum_{i=1}^{n-1}\sum_{j=0}^{i-1}[a^ib^{i-1-j},T]^{\tau^{\left(n-1-\frac{i(i-1)}{2}-j\right)}}+\left(\frac{n}{2}-1\right)T^{-n}+(a^n)^{\tau^{\frac{n}{2}}}+(b^n)^{\tau^{\frac{n}{2}}}.$$

Using the above formulas and the decomposition of $[a^ib^j,T]$ one could, after some computations, derive a basis with of R^{ab}_{Heis} with the conjugates of $(ab)^n$ included as generators, but we will not be performing this task. Notice that, in the even case even though $(ab)^n$ is not in R_{Heis}, we expect from the ramification index over the point ∞ for $(ab)^{2n}$ to be included, which happens as

$$(ab)^nT^{-\frac{n}{2}} + ((ab)^nT^{-\frac{n}{2}})^{\tau^{\frac{n}{2}}} = (ab)^{2n},$$

and this is stabilized under the action of $\tau^{\frac{n}{2}}$, which implies we have the elements $((ab)^{2n})^{\alpha^i\tau^k}$, $0\leq i\leq n-1, 0\leq k\leq \frac{n}{2}-1$ in R_{Heis_n} for even n.

2.5. **Elements stabilized.** In this section we will partially describe the fundamental group of the open Heisenberg curve as

$$R_{\mathrm{Heis}_n} = \langle a_1,b_1,\ldots,a_g,b_g,\gamma_1,\ldots,\gamma_m \mid \gamma_1\gamma_2\cdots\gamma_m\cdot[a_1,b_1][a_2,b_2]\cdots[a_g,b_g]=1\rangle,$$

where g is the genus of the Heisenberg curve and m is the number of branched points of the cover. Combining the Schreier index formula with the above description we have

$$2g+m-1 = n^3+1,$$

from which we expect that m is $3n^2$ in the unramified case and $\frac{5}{2}n^2$ in the ramified case. The decomposition of $(ab)^n$ (or $(ab)^nT^{\frac{n}{2}}$ accordingly) suggests that some free generators $[a^ib^j,T]$, or equivalently some conjugates of $[a,T],[b,T]$ can be swapped with the n^2-1 conjugates of $(ab)^n$ (or accordingly with the conjugates of $(ab)^nT^{\frac{n}{2}}$, and in a second base change with the $\frac{n^2}{2}-1$ conjugates of $(ab)^{2n}$). Thus, the above presentation of R_{Heis} can be realized with a_i,b_i being the elements $[a,T]^{\alpha^i\tau^k},[b,T]^{\alpha^i\beta^j\tau^k}$ for all the indices from proposition 13 that remain after the supposed base change, and the elements γ_i with their stabilizers are depicted in the following table:

Invariant element γ_m	Index m	Fixed by
Unramified case $(n,2)=1$:		
$(a^n)^{\tau^k\beta^i}$, $1\leq k,i\leq n$	$1\leq m\leq n^2$	$\langle\alpha\tau^i\rangle$
$(b^n)^{\alpha^i\tau^k}$, $1\leq k,i\leq n$	$n^2+1\leq m\leq 2n^2$	$\langle\beta\tau^{n-i}\rangle$
$((ab)^n)^{\alpha^i\tau^k}$, $1\leq k,i\leq n$	$2n^2+1\leq m\leq 3n^2$	$\langle\alpha\beta\tau^{n-i}\rangle$
Ramified case $(n,2)=2$:		
$(a^n)^{\tau^k\beta^i}$, $1\leq k,i\leq n$	$1\leq m\leq n^2$	$\langle\alpha\tau^i\rangle$
$(b^n)^{\alpha^i\tau^k}$, $1\leq k,i\leq n$	$n^2+1\leq m\leq 2n^2$	$\langle\beta\tau^{n-i}\rangle$

Invariant element γ_m	Index m	Fixed by
Unramified case $(n,2)=1$:		
$((ab)^{2n})^{\alpha^i\tau^k}$, $_{1\leq k\leq \frac{n}{2}}^{1\leq i\leq n}$	$2n^2+1\leq m\leq \frac{5}{2}n^2$	$\langle \alpha\beta\tau^{n-i}\rangle, \langle \tau^{\frac{n}{2}}\rangle$

Definition 15. Let Γ be the free $\mathbb{Z}$-module generated by $\langle \gamma_1, \dots \gamma_{3n^2}\rangle$ in the unramified case or $\langle \gamma_1, \dots, \gamma_{\frac{5}{2}n^2}\rangle$ in the ramified case. Equivalently, we can set Γ to be $R_{\mathrm{Heis}} \cap \langle a^n, b^n, (ab)^n\rangle$ in both cases.

We denote by X_H the Heisenberg curve as the ramified cover $X_H \to \mathbb{P}^1$ and we have that the homology group of the curve is

$$H_1(X_H, \mathbb{Z}) = \frac{R_{\mathrm{Heis}}/R'_{\mathrm{Heis}}}{\Gamma}.$$

Having the $\mathbb{Z}[H_n]$-basis of R^{ab}_{Heis} in mind, the natural question now is if both $[a,T]$ and $[b,T]$ generate the homology as a $\mathbb{Z}[H_n]$-module. Although the decomposition of $(ab)^n$ is very complicated and contains many non-free elements, we can extract from it enough information to asnwer this.

Proposition 16. *For $n=3$ the homology of the Heisenberg curve is generated by $[a,T]$ as a $\mathbb{Z}[H_3]$-module. For $n \geq 4$ it is generated by both $[a,T], [b,T]$ as a $\mathbb{Z}[H_n]$-module, this means they belong in different classes* $\mathrm{mod}\Gamma$.

Proof. Observe in the decomposition of $(ab)^n$ that for $n=3$ the only non-zero power of b is 1, which means only one conjugate of $[b,T]$ appears, say $[b,T]^x$. We can apply the action of x^{-1} such that $[b,T]$ appears and by reducing $\mathrm{mod}\Gamma$ every other element is of the form $[a,T]^y$, that is $[b,T]$ is linearly dependant on conjugates of $[a,T]$ modulo Γ. The same reasoning when $n \geq 4$ provides that $[b,T]$ will be linearly dependant to conjugates of $[a,T]$ and to at least one of its own conjugates modulo Γ. □

We provide the exact computation when $n=3$, combining the decompositions of $(ab)^n, [a^i b^j, T]$ and T^n we are left with

$$(ab)^n = -[a,T]^{\tau^2} - [b,T]^{\alpha^2\tau} - [a,T]^{\alpha\tau} - (a^3)^{\beta} + \beta^3$$

and applying the action of $\alpha\tau^2$ and reducing $\mathrm{mod}\Gamma$ we get

$$[b,T] \equiv -[a,T]^{\alpha\tau} - [a,T]^{\alpha^2} \mathrm{mod}\Gamma.$$

2.6. **Braid group action.** In this section we will describe the action of the braid group on $H_1(X_H, \mathbb{Z})$. The braid group B_3 in this setting will be realized as an automorphism group of F_2 in terms of the faithfull Artin representation. See [12, 2.1] for a more detailed description and an application to cyclic covers of the projective line, also [11] for the action of B_3 on the closed Fermat surface. The group B_3 is generated by σ_1 and σ_2 such that

$$\sigma_1(a) = aba^{-1} \quad \sigma_1(b) = a \quad \sigma_2(a) = a \quad \sigma_2(b) = a^{-1}b^{-1}.$$

We would like to have a well-defined action of B_3 on the fundamental group of the Heisenberg curve, as it happens with the Fermat curve, however there is an obstruction.

Lemma 17. *The group R_{Heis_n} is characteristic for odd n, which means every automorphism in* $\mathrm{Aut}(F_2)$ *keeps R_{Heis_n} invariant.*

Proof. The group $R_{\mathrm{Heis}} \subset F_2 = \langle a, b\rangle$ is the normal closure of the elements $a^n, b^n, [a,T], [b,T]$ and the automorphism group of F_2 is generated by the Nielsen transformations below, see [4, Th. 1.5, p.125]

$$\mathrm{A}ut(F_2) = \langle n_a, n_b, n_{ab}, n_{ba}\rangle$$

where

$$\begin{array}{cccc} n_a(a) = a^{-1} & n_a(b) = b & n_b(a) = a & n_b(b) = b^{-1} \\ n_{ab}(a) = ab & n_{ab}(b) = b & n_{ba}(a) = a & n_{ba}(b) = ba \end{array}$$

We compute

$$\begin{aligned} n_a(a^n) &= a^{-n}, \\ n_a([a,T]) &= [a^{-1}, [a^{-1}, b]] = [a,T]^{a^{-2}T^{-1}}, \\ n_a([b,T]) &= [b, [a^{-1}, b]] = [b, [b,a]^{a^{-1}}] = [T,b]^{a^{-1}}, \\ n_{ab}(a^n) &= (ab)^n \in R_{\mathrm{Heis}_n} \text{ in the unramified case}, \\ n_{ab}([a,T]) &= [ab, [ab, b]] = [ab, [a,b]] = [b,T]^a \cdot [a,T], \\ n_{ab}([b,T]) &= [b, [ab, b]] = [b,T]. \end{aligned}$$

and the action of n_b, n_{ba} is symmetrical. □

Notice that because of the anomaly in $n_{ab}(a^n)$, R_{Heis_n} cannot be characteristic for even n. In contrast, in the unramified case the previous lemma tells us that the braid group B_3 keeps the Heisenberg curve invariant, this means it has a well-defined action on R_{Heis} which induces an action on $H_1(X_H, \mathbb{Z})$.

Remark 18. There are two things worth pointing out about the ramified case here, one is that B_3 keeps Γ invariant and thus there is a well-defined action of B_3 on $H_1(X_H, \mathbb{Z})$ nonetheless. The second thing is that since $\sigma_2(b^n) = ((ab)^{-n})^b \notin R_{\mathrm{Heis}}$ the braid group B_3 sends the Heisenberg curve to its conjugate curves defined by the orbit of $\sigma_2(R_{\mathrm{Heis}})$. This is an interesting example in the field of moduli versus field of definition perspective, as discussed in [5], [6], where for a base field K a curve X defined a priori on a separable closure K_s might or might not be definable over K. For this question the isomorphic curves of X with different models than X, after an action of an automorphism, are taken into account.

To expand further on the previous remark, suppose that we work over $\overline{\mathbb{Q}}$ for simplicity. Following the definitions in [5], we have that $\Pi_{\overline{\mathbb{Q}}}(B^*)$ is the geometric fundamental group of $B^* = \mathbb{P}^1_{\overline{\mathbb{Q}}} - \{0, 1, \infty\}$. The group $\Pi_{\overline{\mathbb{Q}}}(B^*)$ is canonically isomorphic to the profinite free group $\mathfrak{F}_2$ and fits the following split exact sequence

$$1 \to \Pi_{\overline{\mathbb{Q}}}(B^*) \to \Pi_{\mathbb{Q}}(B^*) \to \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to 1,$$

with the middle term $\Pi_{\mathbb{Q}}(B^*)$ being the arithmetic fundamental group of B^*. Through Galois theory, G-covers of B^* correspond to surjective homomorphims $\Psi : \Pi_{\mathbb{Q}}(B^*) \to G$ and also the well-defined action up to inner automorphisms and sections of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\Pi_{\overline{\mathbb{Q}}}(B^*)$ induces an action on these covers.

As the braid group B_3 can act on the topological generators of $\mathfrak{F}_2$ it has a similar action on the covers. Let σ denote $\sigma_2 \in B_3$ and consider the homomorphisms

$$\Psi, \Psi^\sigma : \mathfrak{F}_2 \to H_n,$$

corresponding to quotiening by the profinite groups $\hat{R}_{\mathrm{Heis}_n}, \sigma(\hat{R}_{\mathrm{Heis}_n})$ respectively. We provide an interesting fact above these two covers in the next proposition.

Proposition 19. *Assume n to be even. The Heisenberg curve and its σ-conjugate are not isomorphic over $\overline{\mathbb{Q}}$.*

Proof. According to [5], the curves are isomorphic if and only if Ψ, Ψ^σ have conjugate images, that is there is an element h in H_n such that

$$\Psi(x) = h\Psi^\sigma(x)h^{-1}, \text{ for all } x \in \Pi_{\overline{\mathbb{Q}}}(B^*).$$

Set x to be ab, then $\Psi(ab) = \alpha\beta$ which is of order $2n$ in H_n, however, $\Psi^\sigma(ab)$ is of order n since $(ab)^n$ is contained in $\sigma_2(R_{\mathrm{Heis}_n})$. Thus the above criterion is not satisfied and the result follows. □

We describe now the action of B_3 on the $\mathbb{Z}[H_n]$-generators of $H_1(X_H, \mathbb{Z})$ according to proposition 16, which will be useful to describe the Burau representation. Firstly, we have that

$$\sigma_1(T) = (T^{-1})^a, \quad \sigma_2(T) = [b^{-1}, a^{-1}] = (T^{-1})^{b^{-1}a^{-1}},$$

and with a repeated use of lemma 6 we compute the following.

$$\begin{aligned}
\sigma_1[a,T] &= -[b,T]^{\alpha\tau^{-1}} \\
\sigma_1[b,T] &= -[a,T]^{\alpha\tau^{-1}} \\
\sigma_2[a,T] &= [b,T]^{\beta^{-1}\tau^{-1} - \alpha^{-1}\beta^{-1}\tau^{-1}} - [a,T]^{\alpha^{-1}\tau^{-1}} \\
\sigma_2[b,T] &= [b,T]^{(\alpha^{-1}\beta^{-1})^2\tau^{-1}} + [a,T]^{\alpha^{-1}\beta^{-1}\alpha^{-1}\tau^{-1}}.
\end{aligned}$$

2.6.1. *Burau Representation.* We will see that the action of B_3 on the generators of the homology of the Heisenberg curve induces a representation

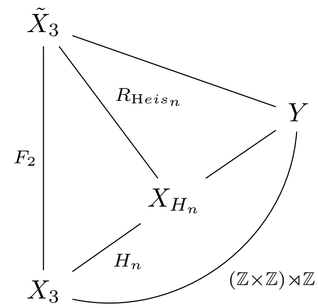
$$\rho : B_3 \longrightarrow \mathrm{GL}(2, A), \tag{6}$$

where

$$A = \mathbb{Z}\langle a^{\pm 1}, b^{\pm 1}\rangle / \langle [a,t], [b,t]), t = [a,b]\rangle$$

and $\mathbb{Z}\langle a^{\pm 1}, b^{\pm 1}\rangle$ is the noncommutative Laurent polynomial ring in two variables over $\mathbb{Z}$. For a similar construction of the Burau representation for cyclic covers of $\mathbb{P}^1$ see [12].

Let $X_3 = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and denote by $\tilde{X}_3$ its universal covering space. We can obtain the Heisenberg curves X_{H_n} as quotients of an abstract curve Y defined as $\tilde{X}_3/I$ where I corresponds to $\langle [a,t], [b,t]\rangle$. Then X_{H_n} is the quotient Y/J, where J corresponds to $\langle a^n, b^n\rangle$ as depicted in the following diagram.



The homology group $H_1(Y, \mathbb{Z})$ through similar techniques from this paper has a conjugation action by the discrete Heisenberg group

$$H = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z} \right\} \cong (\mathbb{Z} \times \mathbb{Z}) \rtimes \mathbb{Z} = \langle a, t \rangle \rtimes \langle b \rangle$$

and is thus an A-module. Repeating the braid action computations from the previous section we get that this action introduces the representation ρ given in eq. (6) which is given by

$$\rho(\sigma_1) = \begin{pmatrix} 0 & -at^{-1} \\ -at^{-1} & 0 \end{pmatrix},$$
$$\rho(\sigma_2) = \begin{pmatrix} -a^{-1}t^{-1} & -a^{-1}b^{-1}a^{-1}t^{-1} \\ (1-a^{-1})b^{-1}t^{-1} & (a^{-1}b^{-1})^2 t^{-1} \end{pmatrix}.$$

Similar to [12] we define

$$\mathbb{Z}_\ell[\overline{H}] = \varprojlim_n \mathbb{Z}_\ell[H_{\ell^n}].$$

Then, we have that

$$H_1(Y, \mathbb{Z}_\ell) = \hat{R}_{\mathrm{Heis}}/\hat{R}'_{\mathrm{Heis}},$$

where R_{Heis} is the fundamental group of Y and $\hat{R}_{\mathrm{Heis}_n}$ is its pro-ℓ completion, which is given by

Proposition 20. *The group $H_1(Y, \mathbb{Z}_\ell)$ is generated by both $[a, t], [b, t]$ as a $\mathbb{Z}_\ell[\bar{H}]$-module.*

Therefore, each element $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $H_1(Y, \mathbb{Z}_\ell)$, using Ihara's action given in eq. (1), on the generators of $H_1(Y, \mathbb{Z}_\ell)$ given in proposition 20.

3. Alexander modules

Let $\mathbb{F}$ be a field of characteristic 0 containing the n different n-th roots of unity. In this section we will use the theory of Alexander modules and the Crowell exact sequence, as described in Chapter 9 from [13], to describe the homology $H_1(X_H, \mathbb{Z})$ of the closed Heisenberg curve as an $\mathbb{F}[H_n]$-module in terms of the characters of the Heisenberg group.

Let F_2 be the free group $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, x_0)$ as before with generators a, b and R_{Heis} as defined previously. Let Γ be $\langle a^n, b^n, (ab)^n \rangle$, we can describe $G := F_2/R_{\mathrm{Heis}} \cap \Gamma$ as

$$G = \langle a, b, ab \mid \ a^{e_1} = b^{e_2} = (ab)^{e_3} = a \cdot b \cdot (ab)^{-1} = 1 \rangle.$$

where $e_1 = e_2 = n$ and $e_3 = n$ or $2n$ if n is odd or even respectively. Let $\overline{R}_{\mathrm{Heis}} := R_{\mathrm{Heis}}/R_{\mathrm{Heis}} \cap \Gamma \cong R_{\mathrm{Heis}} \cdot \Gamma/\Gamma$. This reduces to R_{Heis}/Γ in the unramified case. We have a quotient map

$$\psi : F_2/\Gamma \to F_2/R_{\mathrm{Heis}} \cong H_n.$$

Set also $\varepsilon : \mathbb{Z}[H_n] \to \mathbb{Z}$ to be the augmentation map $\sum a_g g \mapsto \sum a_g$.

We consider $\mathscr{A}_\psi$ to be the *Alexander module*, a free $\mathbb{Z}$-module

$$\mathscr{A}_\psi = \left(\bigoplus_{g \in F_2/\Gamma} \mathbb{Z}[H_n] dg \right) / \langle d(g_1 g_2) - dg_1 - \psi(g_1) dg_2 : \ g_1, g_2 \in F_2/\Gamma \rangle_{\mathbb{Z}[H_n]}$$

where $\langle \cdots \rangle_{\mathbb{Z}H_n}$ is considered to be a $\mathbb{Z}[H_n]$ module generated by the elements appearing inside.

By the above definitions, $\overline{R}^{ab}_{\mathrm{Heis}}$ is $H_1(X_H, \mathbb{Z})$. Define the map $\theta_1 : \overline{R}^{ab}_{\mathrm{Heis}} \to \mathscr{A}_\psi$ given by

$$\overline{R}^{ab}_{\mathrm{Heis}} \ni n \mapsto dn$$

and the map $\theta_2 : \mathscr{A}_\psi \to \mathbb{Z}[H_n]$ to be the homomorphism induced by

$$dg \mapsto \psi(g) - 1 \quad \text{for } g \in G.$$

We have the following exact sequence

$$1 \longrightarrow \overline{R}_{\mathrm{Heis}} \longrightarrow G \xrightarrow{\psi} H_n \longrightarrow 1$$

from which we obtain the Crowell exact sequence of $\mathbb{Z}[H_n]$-modules [13, sec. 9.2]

$$1 \longrightarrow \overline{R}^{ab}_{\mathrm{Heis}} = H_1(X_H, \mathbb{Z}) \xrightarrow{\theta_1} \mathscr{A}_\psi \xrightarrow{\theta_2} \mathbb{Z}[H_n] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 1. \tag{7}$$

For the following proposition, let F_3 be the free group generated by x_1, x_2, x_3, we use that there is a natural epimorphism $\pi : F_3 \to G$ mapping $x_1 \mapsto a, x_2 \mapsto b$ and $x_3 \mapsto (ab)^{-1}$.

Proposition 21. *The module $\mathscr{A}_\psi$ admits a free resolution as a $\mathbb{Z}[H_n]$-module:*

$$\mathbb{Z}[H_n]^4 \xrightarrow{Q} \mathbb{Z}[H_n]^3 \longrightarrow \mathscr{A}_\psi \longrightarrow 0 \tag{8}$$

where 4 and 3 appear as the number of relations and generators of G respectively. The map Q is expressed in form of Fox derivatives [2, sec. 3.1], [13, chap. 8] *as follows, let $\beta_1, \beta_2, \beta_3, \beta_4 \in \mathbb{Z}[H_n]$, then*

$$\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix} \mapsto \begin{pmatrix} \psi\pi(\frac{\partial x_1^{e_1}}{\partial x_1}) & \psi\pi(\frac{\partial x_2^{e_2}}{\partial x_1}) & \psi\pi(\frac{\partial x_3^{e_3}}{\partial x_1}) & \psi\pi(\frac{\partial x_1 \cdot x_2 \cdot x_3}{\partial x_1}) \\ \psi\pi(\frac{\partial x_1^{e_1}}{\partial x_2}) & \psi\pi(\frac{\partial x_2^{e_2}}{\partial b}) & \psi\pi(\frac{\partial x_3^{e_3}}{\partial x_2}) & \psi\pi(\frac{\partial x_1 \cdot x_2 \cdot x_3}{\partial x_2}) \\ \psi\pi(\frac{\partial x_1^{e_1}}{\partial x_3}) & \psi\pi(\frac{\partial x_2^{e_2}}{\partial x_3}) & \psi\pi(\frac{\partial x_3^{e_3}}{\partial x_3}) & \psi\pi(\frac{\partial x_1 \cdot x_2 \cdot x_3}{\partial x_3}) \end{pmatrix} \cdot \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix}$$

where π is the natural epimorphism $F_3 \to G$ as defined previously.

Proof. See [13, Cor. 9.6]. □

We apply on the exact sequence 7 and the functor $\otimes_{\mathbb{Z}}\mathbb{F}$ to get the exact sequence of $\mathbb{F}[H_n]$-modules

$$1 \longrightarrow H_1(X_H, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F} \xrightarrow{\theta_1 \otimes 1} \mathscr{A}_\psi \otimes_{\mathbb{Z}} \mathbb{F} \xrightarrow{\theta_2 \otimes 1} \mathbb{F}[H_n] \xrightarrow{\varepsilon} \mathbb{F} \longrightarrow 1 \tag{9}$$

In general, it is well-known that the tensor functor is not left exact, but since $\operatorname{char}\mathbb{F} = 0$ we have that $\mathbb{F}$ is a flat $\mathbb{Z}$-module and this provides the left exactness. By counting dimensions, we have that

$$\dim_{\mathbb{F}} \mathscr{A}_\psi \otimes_{\mathbb{Z}} \mathbb{F} = 2g + n^3 - 1$$

where g is the genus as in lemma 9. We will describe now the regular representation $\mathbb{F}[H_n]$ in terms of the irreducible characters χ_{ijs} of H_n, see appendix A for their definition in a short survey of them. Recall that every irreducible representation χ appears $\deg\chi$ times in the decomposition of $\mathbb{F}[H_n]$.

$$\mathbb{F}[H_n] = \bigoplus_{j=0}^{n-1} \bigoplus_{i,s=0}^{\gcd(n,j)-1} \mathbb{F}\frac{n}{\gcd(n,j)}\chi_{ijs}.$$

The method now is to use the free resolution given in eq. (8) in order to describe $\mathscr{A}_\psi \otimes_{\mathbb{Z}} \mathbb{F}$ as an $\mathbb{F}[H_n]$-module and then use the Crowell exact sequence to understand the homology. The $\mathbb{Z}[H_n]$-module $\mathscr{A}_\psi$ is the cokernel of the map Q. We will denote as previously α, β, τ the images in H_n. We compute

$$\frac{\partial x_i^{e_i}}{\partial x_j} = \delta_{ij}\left(1 + x_i + x_i^2 + \cdots + x_i^{e_i - 1}\right)$$
$$\frac{\partial x_1 \cdot x_2 \cdot x_3}{\partial x_1} = 1$$
$$\frac{\partial x_1 \cdot x_2 \cdot x_3}{\partial x_2} = x_1$$
$$\frac{\partial x_1 \cdot x_2 \cdot x_3}{\partial x_3} = x_1 x_2$$

Set the following:

$$\Sigma_1 = 1 + \alpha + \cdots + \alpha^{n-1}$$
$$\Sigma_2 = 1 + \beta + \cdots + \beta^{n-1}$$
$$\Sigma_3 = 1 + (\alpha\beta) + \cdots + (\alpha\beta)^{n-1}$$

In the ramified case we are interested in having the following sum instead of Σ_3

$$\begin{aligned}\Sigma_3' &= 1 + (\alpha\beta) + \cdots + (\alpha\beta)^{2n-1} = \\ &= 1 + (\alpha\beta) + \cdots (\alpha\beta)^{n-1} + \tau^{\frac{n}{2}} + \tau^{\frac{n}{2}}(\alpha\beta) + \cdots \tau^{\frac{n}{2}}(\alpha\beta)^{n-1} = \\ &= (1 + \tau^{\frac{n}{2}})\Sigma_3.\end{aligned}$$

Set Σ_3^* to be Σ_3 or Σ_3' varying based on the ramification as previously. The map Q in proposition 8 is given by the matrix on the left-hand side of the following equation

$$\begin{pmatrix} \Sigma_1 & 0 & 0 & 1 \\ 0 & \Sigma_2 & 0 & \alpha \\ 0 & 0 & \Sigma_3^* & \alpha\beta \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix} = \begin{pmatrix} \Sigma_1 r_1 + r_4 \\ \Sigma_2 r_2 + \alpha r_4 \\ \Sigma_3^* r_3 + \alpha\beta r_4 \end{pmatrix} \tag{10}$$

where $r_i \in \mathbb{Z}[H_n]$. Observe that

$$\begin{aligned} \Sigma_1 \alpha &= \Sigma_1, \\ \Sigma_2 \beta &= \Sigma_2, \\ \Sigma_3^* \alpha\beta &= \Sigma_3^*, \end{aligned}$$

in particular, in the even n case

$$\Sigma_3' \tau^{\frac{n}{2}} = \Sigma_3'. \tag{11}$$

Lemma 22. *For $i = 1, 2$ the following isomorphisms hold*

$$\Sigma_i \mathbb{Z}[H_n] \cong \Sigma_i \mathbb{Z}[\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}]$$

and

$$\Sigma_3^* \mathbb{Z}[H_n] \cong \begin{cases} \Sigma_3 \mathbb{Z}[\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}], & \text{in the unramified case,} \\ \Sigma_3' \mathbb{Z}[\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/\frac{n}{2}\mathbb{Z}], & \text{in the ramified case.} \end{cases}$$

Proof. We cannot work as in the proof of lemma 4.4 in [11], because H_n is not a direct product of groups. We will, however, determine the action explicitly. Let $\beta^j \alpha^i \tau^k$ be an element in H_n. Then

$$\begin{aligned} \Sigma_1 \beta^j \alpha^i \tau^k &= \Sigma_1 \alpha^i \beta^j \tau^{k-ij} = \Sigma_1 \beta^j \tau^{k-ij} \in \Sigma_1 \mathbb{Z}[\langle\beta\rangle \times \langle\tau\rangle], \\ \Sigma_2 \beta^j \alpha^i \tau^k &= \Sigma_2 \alpha^i \tau^k \in \Sigma_1 \mathbb{Z}[\langle\alpha\rangle \times \langle\tau\rangle]. \end{aligned}$$

For the action of Σ_3^* we have to form pairs of $\alpha\beta$ instead of forcing α^i to commute with β^j, notice that,

$$\Sigma_3^* \alpha = \Sigma_3^* \alpha \beta^n = \Sigma_3^* \beta^{n-1},$$

and

$$\Sigma_3^* \beta = \Sigma_3^* \beta \alpha^n = \Sigma_3^* \alpha^{n-1} \tau^{-1}.$$

These generalize to

$$\begin{aligned} \Sigma_3^* \alpha^i &= \Sigma_3^* \beta^{n-i} \tau^{1+2+\cdots+(i-1)}, \\ \Sigma_3^* \beta^j &= \Sigma_3^* \alpha^{n-j} \tau^{-1-2-\cdots-j}. \end{aligned}$$

Thus, $\Sigma_3^* \mathbb{Z}[\langle\alpha\rangle \times \langle\tau\rangle] \subseteq \Sigma_3^* \mathbb{Z}[\langle\beta\rangle \times \langle\tau\rangle]$ and the opposite inlcusion holds as well. Notice also that by equation 11 only the powers $\tau^i, i = 0, \ldots, \frac{n}{2} - 1$ will survive in the ramified case. The result follows. □

We have that $\mathrm{Im}(Q)$ equals to the space generated by elements

$$\begin{pmatrix} \Sigma_1 r_1 \\ \Sigma_2 r_2 \\ \Sigma_3^* r_3 \end{pmatrix} + \begin{pmatrix} 1 \\ \alpha \\ \alpha\beta \end{pmatrix} r_4.$$

For $r_1, \ldots, r_4 \in \mathbb{Z}[H_n]$ the first summand forms a free $\mathbb{Z}$-module of rank $3n^2$ (resp. $\frac{5}{2}n^2$) for the unramified (resp. ramified) case and the second summand is a free $\mathbb{Z}$-module of rank n^3. Furthermore, their intersection is $\mathbb{Z}$.

Indeed, suppose we have $r_1, \ldots, r_4 \in \mathbb{Z}[H_n]$ such that

$$(\Sigma_1 r_1, \Sigma_2 r_2, \Sigma_3^* r_3) = r_4(1, \alpha, \alpha\beta)$$

then by comparing the coordinates, $r_4 = \Sigma_1 r_1$ which implies α acts trivially on r_4. By the second coordinates, $\Sigma_2 r_2 = \alpha r_4 = r_4$ which implies β acts trivially on r_4. Similarly, by the third coordinates and what we have already established, $\alpha\beta$ acts trivially on r_4. This implies that r_4 is invariant under the action of the whole group H_n, that is r_4 belongs to the rank one $\mathbb{Z}$-module generated by $\Sigma_1\Sigma_2\Sigma_3^*$. We have proved that

Lemma 23.

$$\mathrm{Im}(Q) = \left(\bigoplus_{\nu=1}^{2} \Sigma_i \mathbb{Z}[H_n] \right) \bigoplus \Sigma_3^* \mathbb{Z}[H_n] \bigoplus \mathbb{Z}[H_n]/\mathbb{Z}\Sigma_1\Sigma_2\Sigma_3^*.$$

Also,

$$\mathrm{rank}_{\mathbb{Z}}\, Q = \begin{cases} n^3 + 3n^2 - 1, & \textit{in the unramified case}, \\ n^3 + \frac{5}{2}n^2 - 1, & \textit{in the ramified case}. \end{cases}$$

Remark 24. If Σ is an element invariant by the action of a subgroup $H < G$, then $\Sigma H = \mathrm{Ind}_H^G \mathbb{F}$, where $\mathbb{F}$ has the trivial H action. Indeed, observe that $\mathrm{Ind}_H^G \mathbb{F} = \mathbb{F} \otimes_{\mathbb{F}[H]} \mathbb{F}[G] = \Sigma \mathbb{F}[G]$.

The module $\Sigma_1 \mathbb{F}[H_n]$ is isomorphic to $\Sigma_1 \mathbb{F}[\langle\beta\rangle \times \langle\tau\rangle]$ by the proof of lemma 22. Using remark 24 we see that as a representation its character χ_{Σ_1} is the character of $\mathrm{Ind}_{\langle\alpha\rangle}^{H_n} \mathbb{F}$. Denote by p_α the trivial character of the group $\langle\alpha\rangle$. Using Frobenius reciprocity we compute:

$$\begin{aligned} \langle \chi_{ijs}, \chi_{\Sigma_1} \rangle_{H_n} &= \langle \mathrm{Res}\, \chi_{ijs}, p_\alpha \rangle_{\langle\alpha\rangle} = \frac{1}{n} \sum_{k=0}^{n-1} \chi_{ijs}(\alpha^k) \\ &= \frac{1}{\gcd(n,j)} \sum_{k=0}^{\gcd(n,j)-1} \zeta^{k \frac{n}{\gcd(n,j)} i} = \begin{cases} 1, & i = 0, \\ 0, & i \neq 0. \end{cases} \end{aligned}$$

We have computed that

$$\Sigma_1 \mathbb{F}[H_n] = \bigoplus_{j=0}^{n-1} \bigoplus_{s=0}^{\gcd(n,j)-1} \mathbb{F}\chi_{0js}.$$

Notice that the above consists of the n one-dimensional subspaces ($j = 0$) and of $(n-1) \cdot \gcd(n,j)$ subspaces ($j \neq 0$) of dimension $n/\gcd(n,j)$, adding up to the correct dimension n^2.

A similar computation yields that

$$\Sigma_2 \mathbb{F}[H_n] = \bigoplus_{j=0}^{n-1} \bigoplus_{i=0}^{\gcd(n,j)-1} \mathbb{F}\chi_{ij0}.$$

In the module $\Sigma_3^* \mathbb{F}[H_n]$, we have that $\alpha\beta$ is annihilated, therefore $\Sigma_3^* \mathbb{F}[H_n] = \mathrm{Ind}_{\langle\alpha\beta\rangle}^{H_n} \mathbb{F}$. Moreover, we can expect the one dimensional irreducible characters χ_{i0s} that appear in $\chi_{\Sigma_3^*}$ map $\alpha\beta = \beta\tau\alpha$ to 1, that is $i + s \equiv 0 \bmod n$. We will see how this generalizes to the higher dimensional irreducible characters using Frobenius reciprocity. Let again $p_{\alpha\beta}$ be the trivial character of $\mathbb{F}[\langle\alpha\beta\rangle]$. We set $d_j = \gcd(n, j)$ and use the fact that $(\alpha\beta)^k = \beta^k\alpha^k\tau^{\binom{k+1}{2}}$. We compute

$$\begin{aligned}\langle \chi_{ijs}, \chi_{\Sigma_3^*}\rangle_{H_n} &= \langle \operatorname{Res}\chi_{ijs}, p_{\alpha\beta}\rangle_{\langle\alpha\beta\rangle} = \frac{1}{|\langle\alpha\beta\rangle|}\sum_{k=0}^{|\langle\alpha\beta\rangle|-1}\chi_{ijs}((\alpha\beta)^k)\\ &= \begin{cases}\frac{1}{d_j}\sum\limits_{k=0}^{d_j-1}\zeta^{k\frac{n}{d_j}(i+s)+j\binom{\frac{kn}{d_j}+1}{2}}, & 2\nmid n,\\ \frac{1}{2d_j}\sum\limits_{k=0}^{2d_j-1}\zeta^{k\frac{n}{d_j}(i+s)+j\binom{\frac{kn}{d_j}+1}{2}}, & 2\mid n.\end{cases}\end{aligned}$$

For odd n the quantity $j\binom{\frac{kn}{d_j}+1}{2}$ is divisible by n, thus

$$\langle \chi_{ijs}, \chi_{\Sigma_3}\rangle_{H_n} = \frac{1}{d_j}\sum_{k=0}^{d_j-1}\zeta^{k\frac{n}{d_j}(i+s)} = \begin{cases}1, & i+s\equiv 0 \bmod d_j,\\ 0, & i+s\not\equiv 0 \bmod d_j.\end{cases}$$

We deal with n being even now. Using that $(ab)^k = (ab)^{k'}\tau^{\frac{n}{2}}$ for $k \geq \frac{n}{2}, k \equiv k' \bmod \frac{n}{2}$, we have that

$$\langle \chi_{ijs}, \chi_{\Sigma_3'}\rangle_{H_n} = \frac{1}{2d_j}(1+\zeta^{j\frac{n}{2}})\sum_{k=0}^{d_j-1}\zeta^{k\frac{n}{d_j}(i+s)+j\binom{\frac{kn}{d_j}+1}{2}},$$

which is 0 if j is not even. We thus assume j is even now, it easy to see that

$$j\binom{\frac{kn}{d_j}+1}{2} \equiv \begin{cases}0 \bmod n, & 2\nmid \frac{n}{d_j},\\ k\frac{j}{2}\frac{n}{d_j} \bmod n, & 2\mid \frac{n}{d_j},\end{cases}$$

from which we deduce

$$\langle \chi_{ijs}, \chi_{\Sigma_3'}\rangle_{H_n} = \begin{cases}1, & i+s\equiv 0 \bmod d_j,\ 2\nmid\frac{n}{d_j},\\ 1, & i+s\equiv -\frac{j}{2} \bmod d_j,\ 2\mid\frac{n}{d_j},\\ 0, & \text{otherwise.}\end{cases}$$

Let j' be 0 or $-\frac{j}{2}$ according to the previous statement. We have proved that

$$\Sigma_3^*\mathbb{F}[H_n] = \begin{cases}\bigoplus\limits_{j=0}^{n-1}\ \bigoplus\limits_{\substack{i,s=0\\ i+s\equiv 0 \bmod \gcd(n,j)}}^{\gcd(n,j)-1}\mathbb{F}\chi_{ijs}, & 2\nmid n,\\ \bigoplus\limits_{\substack{j=0\\ 2\mid j}}^{n-1}\ \bigoplus\limits_{\substack{i,s=0\\ i+s\equiv j' \bmod \gcd(n,j)}}^{\gcd(n,j)-1}\mathbb{F}\chi_{ijs}, & 2\mid n,\end{cases}$$

adding up to the correct dimensions n^2 and $n^2/2$ respectively.

Additionally, the module $\mathbb{F}[H_n]/\Sigma_1\Sigma_2\Sigma_3^*$ has every possible character χ_{ijs} except for those induced by $\chi_{ij}\otimes\chi_s$ which map all elements $\alpha, \beta, \alpha\beta$ to 1, which only happens for $(i, j, s) = (0, 0, 0)$. More specifically, $\mathbb{F}[H_n]/\Sigma_1\Sigma_2\Sigma_3^*$ has every possible character except for χ_{000}.

Counting all the characters that appear previously, we set $z_j(i, s)$ as the number of times the character χ_{ijs} appears on all $\Sigma_1\mathbb{F}[H_n], \Sigma_2\mathbb{F}[H_n]$ and $\Sigma_3^*\mathbb{F}[H_n]$, that is in compact notation,

$$z_j(i,s) = \begin{cases} [i=0 \text{ or } s=0] + [i=s=0] + [i+s \equiv 0 \bmod \gcd(n,j)], & 2 \nmid n, \\ [i=0 \text{ or } s=0] + [i=s=0] + [i+s \equiv j' \bmod \gcd(n,j)], & 2 \mid n, j \\ [i=0 \text{ or } s=0] + [i=s=0], & 2 \mid n, 2 \nmid j, \end{cases}$$

where $[P]$ is 1 if property P holds and 0 otherwise, with the convention that all equivalences are satisfied $\bmod 1$.

Lemma 25. *We have the following decomposition*

$$\mathrm{Im}(Q) \otimes \mathbb{F} = \bigoplus \mathbb{F} c_{ijs} \chi_{ijs}$$

where

$$c_{ijs} = \begin{cases} \frac{n}{\gcd(n,j)} + z_j(i,s), & \text{if } (i,j,s) \neq (0,0,0), \\ \frac{n}{\gcd(n,j)} - 1 + z_j(i,s) = 3, & \text{if } (i,j,s) = (0,0,0). \end{cases}$$

Lemma 26. *We have the decomposition of the Alexander module*

$$\mathscr{A}_\psi \otimes \mathbb{F} = \bigoplus_{j=0}^{n-1} \bigoplus_{i,s=0}^{\gcd(n,j)-1} \mathbb{F} a_{ijs} \chi_{ijs},$$

where

$$a_{ijs} = \left(3 \frac{n}{\gcd(n,j)} - c_{ijs} \right).$$

Also,

$$\mathrm{rank}_{\mathbb{Z}} \mathscr{A}_\psi = \begin{cases} 2n^3 - 3n^2 + 1, & \text{in the unramified case,} \\ 2n^3 - \frac{5}{2}n^2 + 1, & \text{in the ramified case,} \end{cases}$$

agreeing with $2g + n^3 - 1$ that we have already counted from the Crowell exact sequence.

Proof. The Alexander module $\mathscr{A}_\psi$ is the cokernel of Q, thus

$$\begin{aligned} \mathrm{rank}_{\mathbb{Z}} \mathscr{A}_\psi &= \mathrm{rank}_{\mathbb{Z}} \mathbb{Z}[H_n]^3 - \mathrm{rank}_{\mathbb{Z}} Q \\ &= 3n^3 - \begin{cases} n^3 + 3n^2 - 1, \\ n^3 + \frac{5}{2}n^2 - 1, \end{cases} . \\ &= \begin{cases} 2n^3 - 3n^2 + 1, & \text{in the unramified case,} \\ 2n^3 - \frac{5}{2}n^2 + 1, & \text{in the ramified case,} \end{cases} . \end{aligned}$$

and the decomposition of $\mathscr{A}_\psi \otimes \mathbb{F}$ follows from the decomposition of $\mathbb{F}[H_n]$ into characters. □

Theorem 27.

$$H_1(X_H, \mathbb{F}) = \bigoplus_{j=0}^{n-1} \bigoplus_{i,s=0}^{\gcd(n,j)-1} \mathbb{F} h_{ijs} \chi_{ijs},$$

where

$$h_{ijs} = \begin{cases} \frac{n}{\gcd(n,j)} - z_j(i,s), & \text{if } (i,j,s) \neq (0,0,0), \\ 0, & \text{if } (i,j,s) = (0,0,0). \end{cases} \tag{12}$$

Proof. Follows from the lemma 26 and the Crowell exact sequence as in (9). More specifically, for every non-principal character we have that

$$h_{ijs} = a_{ijs} - \frac{n}{\gcd(n,j)} = 2\frac{n}{\gcd(n,j)} - c_{ijs} = \frac{n}{\gcd(n,j)} - z_j(i,s).$$

□

Appendix A.
Irreducible Representations of H_n

In this section we will compute the irreducible representations of the Heisenberg group H_n. This is by no means a new result and it can be found for instance in the work of J. Grassberger and G. Hörmann in [8]. We rely on the general method regarding semi-direct products of groups with the normal group being abelian, as it is described by Serre in section 8.2 of his book [14].

Indeed, we can realize H_n as the semi direct product $\langle \alpha, \tau \rangle \rtimes \langle \beta \rangle$ since every element of H_n can be expressed as $\beta^j \tau^k \alpha^i$, $0 \leq i, j, k \leq n-1$ with $\tau = [\alpha, \beta]$ commuting with both α and β. As H_n is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^2 \rtimes \mathbb{Z}/n\mathbb{Z}$ we denote by χ_{ij} and χ_s the irreducible characters of the first and the second component respectively, such that

$$\chi_{ij}(\alpha) = \zeta^i,\ \chi_{ij}(\tau) = \zeta^j,\ \chi_s(\beta) = \zeta^s, \quad 0 \leq i, j, s \leq n-1,$$

for a fixed primitive n-th root of unity ζ.

Applying the method in Serre's book, we can conclude that the irreducible representations of H_n are those induced by the product $\chi_{ij} \otimes \chi_s$ for the indices $0 \leq i, s \leq \gcd(n,j) - 1$ and $0 \leq j \leq n-1$. We will denote them by χ_{ijs} and for a g in H_n its image is the linear combination of the values of $\chi_{ij} \otimes \chi_s$ over the elements in the intersection of the conjugacy class of g and the group $G_{ij} = \langle \alpha, \tau \rangle \cdot \langle \beta^{\frac{n}{\gcd(n,j)}} \rangle$. We have explicitly

$$\chi_{ijs}(\beta^m \tau^\lambda \alpha^\mu) = \sum_{\substack{\nu=0 \\ \tau^{\lambda+\nu\mu}\beta^m\alpha^\mu \in G_{ij}}}^{n/\gcd(n,j)-1} \chi_{ij}(\tau^{\lambda+\nu\mu}\alpha^\mu) \cdot \chi_s(\beta^m),$$

for $0 \leq j \leq n-1$ and $0 \leq i, s \leq \gcd(n,j) - 1$, with each χ_{ijs} being of dimension $\frac{n}{\gcd(n,j)}$. Set $d_j = \gcd(n,j)$, the characters χ_{ijs} take values as following:

$$\chi_{ijs}(\beta^m \tau^\lambda \alpha^\mu) = \begin{cases} \frac{n}{d_j}\zeta^{\mu i + ms + \lambda j}, & \frac{n}{d_j} \mid m, \mu, \\ 0, & \text{otherwise}, \end{cases}$$

which we use for computations with the Alexander modules.

As the number of irreducible representations is equal to the number of conjugacy classes, we verify the above by counting the conjugacy classes. The conjugacy class of $\beta^j \tau^k \alpha^i$ consists of the elements $\beta^j \tau^{k+iv-ju} \alpha^i$, or in terms of matrices

$$\begin{pmatrix} 1 & -u & uv-z \\ 0 & 1 & -v \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & i & k \\ 0 & 1 & j \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & u & z \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & i & k+iv-ju \\ 0 & 1 & j \\ 0 & 0 & 1 \end{pmatrix}$$

As argued in [8] two elements $\beta^j \tau^k \alpha^i, \beta^{j'} \tau^{k'} \alpha^{i'}$ are in the same class iff $i = i', j = j'$ and $k = k' + iv - ju \bmod n$ which has a solution iff $k = k' \bmod \gcd(i,j,n)$. Thus, each conjugacy class contains exactly one element $\beta^j \tau^k \alpha^i$ with $k < \gcd(n,j,i)$.

These are in total $\sum_{i,j=0}^{n-1} \gcd(n,j,i)$, which is equal to the sum $\sum_{j=0}^{n-1} \gcd(n,j)^2$ we counted according to Serre, as both are equal to $\sum_{d|n} d^2\phi(\frac{n}{d})$, where ϕ is the Euler's totient function.

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